Hamiltonian group actions, moment maps & reduction

Sept 13

- G Lie group Ex: groups of matrices, $T^n = (S^1)^n, \ldots$

  G-action on a manifold M :=

$$G \times M \xrightarrow{\psi} M \quad C^\infty \text{ map}, \quad g\cdot(h\cdot x) = (gh)\cdot x.$$
$$g, x \longmapsto \psi_g(x) = g\cdot x$$

  Induces a map on Lie algebras: $\mathfrak{g} = T_eG \longrightarrow \mathfrak{X}(M)$ vector fields

$$\xi \longmapsto X_\xi$$

$$X_\xi(x) = \frac{d}{dt}_{|t=0}\Big(\underbrace{\exp(t\xi)}_{\text{1-param. subgp in } G \text{ gen}^d \text{ by } \xi}\cdot x\Big)$$

  Lemma: $\xi \mapsto X_\xi$ is a Lie alg. homomorphism, ie. $X_{[\xi,\eta]} = [X_\xi, X_\eta]$.

  (easy to check from def$^{\text{ns}}$, since both brackets are commutators of exponentiated actions)

- Look at actions that preserve a sympl. form on M, ie. $G \to \mathrm{Symp}(M,\omega)$.

  e.g: symplectic action of $U(1) = S^1$ on (M,ω)

  $\iff$ sympl. v.f. on M, X, s.t. flow of X is $\frac{1?}{2\pi?}$ periodic.

  ($\Rightarrow$: let $X = X_{\frac{\partial}{\partial\theta}}$; $\Leftarrow$: e^{it} acts by $\exp(tX)$).

  Say an S^1-action is Hamiltonian if $X = X_{\frac{\partial}{\partial\theta}}$ is Hamiltonian ($i_X\omega = dH$).

  Similarly, a $T^n = (S^1)^n$-action is Hamiltonian if each S^1-factor acts Hamiltonianly, ie. $\exists$ n functions $H_1 \ldots H_n$ s.t. $X_{\frac{\partial}{\partial\theta_i}} = X_{H_i}$.

- More generally: moment map: package Hamiltonians for all elts of $\mathfrak{g}$!

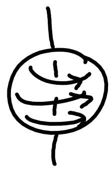
  Def: An action of G on (M,ω) is Hamiltonian if $\exists$ moment map $\mu: M \to \mathfrak{g}^*$ (dual of $\mathfrak{g}$) s.t.
  1) $\forall \xi \in \mathfrak{g}$: let $H_\xi = \langle \mu, \xi\rangle : M \to \mathbb{R}$, X_ξ = v.f. gen^d by ξ. Then $i_{X_\xi}\omega = dH_\xi$.
  2) μ is equivariant wrt G-action, ie.

$$\langle \mu(g\cdot x), \mathrm{Ad}_g(\xi)\rangle = \langle \mu(x), \xi\rangle \quad \forall x \in M, \xi \in \mathfrak{g}, g \in G.$$

★ Case $G = S^1$: $\mu : M \to \mathfrak{g}^* \simeq \mathbb{R}$ is the Hamiltonian for the v.f. generating the action

2) says μ is invt $\mu(g.x) = \mu(x)$ (G is abelian $\Rightarrow$ Ad = trivial)

— true since Ham. flow is tangent to level sets of the Ham. function.



Ex: S^1-action by rotation on S^2 - we've seen Hamiltonian $H = z$.

★ Case $G = T^n$: $\mu = (H_1 \dots H_n) : M \to \mathfrak{g}^* \simeq \mathbb{R}^n$ n Hamiltonians for n circle actions

2) again says μ is invariant under G-action - ie. flow of each X_{H_j} preserves all H_i's!

Claim: this is automatic!

G abelian $\Rightarrow$ $[\cdot,\cdot] = 0$ on $\mathfrak{g}$

$\underset{\text{Lemma}}{\Rightarrow}$ the vf's $X_1 \dots X_n$ generating the action satisfy $[X_i, X_j] = 0$ $\forall i,j$

Recall Poisson bracket $\{f,g\} = dg(X_f) = \omega(X_f, X_g)$.

Fact: $[X_f, X_g] = X_{\{f,g\}}$. Here $[X_i, X_j] = 0 \Rightarrow \{H_i, H_j\} = \text{const}$, actually 0.

ie. $\underline{dH_j(X_{H_i}) = \omega(X_{H_i}, X_{H_j}) = 0}$. (commuting Hamiltonians)

↳ this says flow of X_{H_j} preserves H_i $\Rightarrow$ μ is G-invariant ✓

Also note: for G abelian (torus),

- we've computed $\omega(X_i, X_j) = \{H_i, H_j\} = 0$, so orbits are isotropic!
- $\ker d\mu = \bigcap \ker dH_i = \operatorname{span}(X_i)^{\perp\omega} \supset \operatorname{span}(X_i)$, so regular levels $\mu^{-1}(c)$ are coisotropic.

Ex: T^n-action on $(\mathbb{C}^n, \omega_0)$: $(e^{i\theta_1} \dots e^{i\theta_n})(z_1, \dots z_n) = (e^{i\theta_1} z_1, \dots e^{i\theta_n} z_n)$

$$X_j(z_1 \dots z_n) = (0, \dots, 0, iz_j, 0, \dots, 0) = X_{H_j} \text{ for } H_j = \tfrac{1}{2}|z_j|^2$$

so $$\mu(z_1 \dots z_n) = \left(\tfrac{1}{2}|z_1|^2, \dots, \tfrac{1}{2}|z_n|^2\right).$$

- Toric manifold (symplectically) := (M^{2n}, ω) with Ham. action of T^n.

Then $\mu : M \to \mathbb{R}^n$, regular level sets are Lagrangian (coisotropic, dim. n!) and correspond to where action is free (or has discrete stabilizers; but then can show action not effective). ($d\mu$ onto $\Leftrightarrow$ $X_1 \dots X_n \neq 0$ and linearly independent)

So regular levels = $\bigsqcup$ orbits, and in fact can show connectedness $\Rightarrow$ each nonempty fiber of μ = a single orbit (T^n above regular value, smaller above crit. values) ③

Thm (Atiyah, Guillemin–Sternberg 1982) [actually true for any dim. torus actions]

> (M,ω) compact connected symplectic manifold $\circlearrowleft T^k$ Hamiltonian action, $\mu: M \to \mathbb{R}^k$
> Then the level sets of μ are connected, and
> the image of μ is a <u>convex polytope</u> = Convex Hull (μ(fixed points)).
> + faces have rational slopes.

(Further refined by Delzant: complete description).

Sept 18

<u>Ideas</u> • p fixed point of T^k-action (or of a subtorus) $\Longleftrightarrow$ p critical pt of μ (or of a linear proj. of μ)

Then T^k acts on (T_pM, ω_p). Fact: by averaging, $\exists$ invariant metric and compat. complex structure J_0
generators: k commuting matrices in $sp(2n,\mathbb{R})$, in fact $\in u(n)$.

Then: can diagonalize simultaneously over $\mathbb{C}$ and the eigenspaces are mutually orthogonal symplectic vector subspaces.

ie. block decomposition $\left(\dfrac{\partial}{\partial\theta_j}\right)^{\#} = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & -\lambda_i^j \\ \lambda_i^j & 0 \end{pmatrix}$, $\lambda_i^j \in \mathbb{Z}$ ← so it's 2π-periodic. $\vec{\lambda_i} \in \mathbb{Z}^k$ weights of action.

Moreover, $\bigcap_j \ker(\ldots) = T_p(\text{fixed point set})$ — linearization theorem: using exp chart for invariant metric, the action maps geodesics to geodesics hence it's linear!

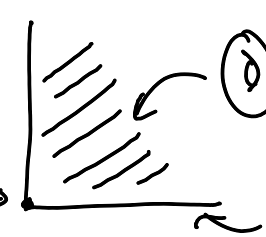
The local moment map is $\mu = \mu(p) + \sum_{i=1}^{n} \frac{1}{2} |z_i|^2 \vec{\lambda_i}$.

• Then critical points of any $\mathbb{R}$ projection of μ are Morse-Bott of <u>even</u> index.
Sanity check: index 1 saddle can't generate a S^1-action: flow not 2π-periodic near crit pt !!!
$\rightarrow$ this implies connectedness of level sets of μ (given that M connected).

• This also implies local convexity of $\mathrm{im}(\mu)$ at critical points
near a fixed point the image looks like cone spanned by $\vec{\lambda_i}$ ($\simeq$ orthant in $\mathbb{R}^k$ in toric case $n=k$)
In general, $\mathrm{Im}(d_p\mu)^{\perp} = \mathfrak{s}$ = Lie algebra of $\mathrm{Stab}(p) \subset T^k$.
so: $d\mu$ onto on directions $\perp \mathfrak{s}$, and projection of μ onto $\mathfrak{s}^*$ is as above.

• The global convexity then follows from connectedness & local convexity.
(argument is by induction on k...)

• <u>Ex</u>: $\mathbb{C}^2$

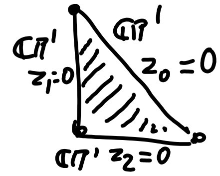


$\mathring{\emptyset}$ product tori (T^2-orbits)

origin = fixed point

on coord. axis, $\emptyset$ (orbits $\simeq S^1$: one factor acts trivially)

- Ex: $\mathbb{CP}^2 = \mathbb{C}^2 \cup \{\text{line at } \infty\}$: $= \{(z_0 : z_1 : z_2)\}/\sim \ \circlearrowleft T^2$



edges = coord. axes $\simeq \mathbb{CP}^1 = S^2$

- Preimage of $\text{int}(\Delta)$ (open stratum) always diffeo to $(\mathbb{C}^*)^n$; can make sense of this holomorphically — to an algebraic geometer, toric var = (partial) compactif^{n} of $(\mathbb{C}^*)^n$

Delzant's thm: Compact toric symplectic (M^{2n}, ω) up to T^n-equivariant symplectomorphism
$\underset{1-1}{\longleftrightarrow}$ Delzant polytopes $\Delta = \text{im}(\mu) \subset \mathbb{R}^{2n}$ (up to translation)
↳ := convex polytope which is
- simple: n edges at each vertex (loc. $\sim \mathbb{R}^n_{\geq 0}$)
- rational: normals to facets $\in \mathbb{Z}^n$ ($\Leftrightarrow$ vectors along edges $\in \mathbb{Z}^n$)
- smooth: integer normals (or edge vectors) at a vertex form a basis of $\mathbb{Z}^n$.

NB: • relabelling basis of $T^n \Leftrightarrow$ action of $GL(n,\mathbb{Z})$ on $\mathfrak{t}^* = \mathbb{R}^n$. Eg.

$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

- $M \rightsquigarrow \Delta$ moment map image, it's Delzant by local behavior of moment map near a fixed point.
- $\Delta \rightsquigarrow M$: one construction is to view $\Delta = \mathbb{R}^N_{\geq 0} \cap$ (affine subspace of dim n) where N = number of facets of Δ ($\exists$ explicit construction: if facets have equations $\ell_j(\xi) \geq 0$, ℓ_j affine linear, $j = 1..N$ then aff. subspace = all linear dependencies among ℓ_j's.
Then construct M by toric reduction of $\mathbb{C}^N$ with respect to action of an $(N-n)$-dim^l subtorus of T^N; reduced space carries residual T^n-action.

- idea for uniqueness: M_1, M_2 with same $\Delta \Rightarrow$ build T^n-equivariant map by taking T^n-orbits $\mu_1^{-1}(c) \mapsto \mu_2^{-1}(c)$ (fixing an origin in each orbit by choosing a Lagrangian section of μ).
Since $\omega = \sum_{i=1}^n dp_i \wedge d\theta_i$ in local coords. (no $d\theta_i \wedge d\theta_j$ by choosing a Lag. ref. section) this is a symplectomorphism.

Symplectic Reduction (Marsden-Weinstein)

Sept 20

Thm: $G \circlearrowright (M,\omega)$ Ham. action of a <u>compact</u> Lie group G, with moment map $\mu: M \to \mathfrak{g}^*$
Assume G acts freely on $\mu^{-1}(0)$ (ie. $\text{Stab}(x) = \{e\}$ $\forall x$)
(NB: equiv$^{ce} \Rightarrow \mu^{-1}(0)$ is preserved by G).

Thm cont'd: Then $\mu^{-1}(0)/G =: M_{red}$ is a smooth mfld and carries a natural sympl. form ω_{red} st. $\pi^*\omega_{red} = \omega|_{\mu^{-1}(0)}$.

$\text{proj}^n\ \mu^{-1}(0) \xrightarrow{\pi} \mu^{-1}(0)/G$

Proof: let $\xi_1 \dots \xi_n$ basis of $\mathfrak{g}$; $X_1 \dots X_n$ corresp. vector fields on M; let $x \in \mu^{-1}(0)$

$H_1 \dots H_n$ components of μ ($H_i = \langle \mu, \xi_i \rangle$ Hamiltonian for X_i).

- G acts freely $\Rightarrow X_1, \dots, X_n$ linearly indep^t at x
  (indeed otherwise $\exists \sum \lambda_i X_i = 0 \Rightarrow \exp(t \sum \lambda_i \xi_i) \cdot x = x$).

  $\Rightarrow dH_1, \dots, dH_n$ linearly indep^t at x
  (dH_i = image of X_i under $\omega: TX \to T^*X$).

  $\Rightarrow d\mu$ onto at x. So: 0 regular value $\Rightarrow \mu^{-1}(0)$ smooth

  G compact acts freely $\Rightarrow$ quotient is a manifold ✓

- At x, $T\mu^{-1}(0) = \ker d\mu = \ker dH_1 \cap \dots \cap \ker dH_n = (\text{span}(X_1 \dots X_n))^{\perp\omega}$
  but we know X_i tangent to level set $\mu^{-1}(0)$ ($\mu^{-1}(0)$ is G-invariant by equivariance)

  $\Rightarrow (T\mu^{-1}(0))^{\perp\omega} = \text{span}(X_1 \dots X_n) \subset T\mu^{-1}(0)$:

  $\rightsquigarrow \mu^{-1}(0)$ is <u>coisotropic</u>

  and isotropic foliation $= \text{span}(X_1 \dots X_n) = T_x(G \cdot x)$ orbit

  $\rightarrow$ isotropic leaves are just G-orbits, and the orbit space is $\mu^{-1}(0)/G = M_{red}$.

  $\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle.

  HW1 Problem 1 $\Rightarrow M_{red}$ has a natural induced sympl. form ▲

Rmk: same argument applies to $\mu^{-1}(c)$ $\forall c$ if $G = T^n$

However if G nonabelian, need $c \in$ Center of $\mathfrak{g}^*$

otherwise G doesn't preserve $\mu^{-1}(c)$!!

Ex: diagonal S^1-action on $(\mathbb{C}^n, \omega_0)$: $e^{i\theta} \cdot (z_1, \dots z_n) = (e^{i\theta} z_1, \dots e^{i\theta} z_n)$

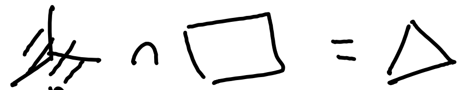
gen^d by $X(z_1 \dots z_n) = (i z_1, \dots i z_n) \Rightarrow \mu = \frac{1}{2} \sum |z_i|^2$.

$\mu^{-1}(0) = \{0\}$ not free (not regular level) ✗

but $\mu^{-1}(\tfrac{1}{2}) = \{ \Sigma |z_i|^2 = 1 \} = S^{2n-1} \subset \mathbb{C}^n$,

$$S^{2n-1}/S^1 \cong (\mathbb{C}^n - \{0\})/\mathbb{C}^*$$
$$\cong \{ \mathbb{C}\text{-lines in } \mathbb{C}^n \} =: \mathbb{CP}^{n-1}$$ is a sympl. mfld!

(in fact carries residual Ham. T^{n-1}-action from the other S^1-actions on $\mathbb{C}^n$!
toric reduction:



$\mathbb{R}^n_{\geq 0}$ $\cap$ $\Sigma \mu_i = 1$ $=$ simplex)

This gives formula for standard Kähler form on $\mathbb{CP}^{n-1}$, and its moment map
$\left(\mu_i = \dfrac{|z_i|^2}{|z_0|^2 + \dots + |z_{n-1}|^2} \quad i = 1 \dots n-1 \right)$.

Ex: $G(k,n)$ = Grassmannian of complex k-planes in $\mathbb{C}^n = \mathrm{Mat}_{k\times n}/GL(k,\mathbb{C})$

$\mathrm{Mat}_{k\times n} = \mathbb{C}^{k\times n}, \omega_0$ has Hamiltonian action of $U(k)$ by left multiplication.

Moment map: $\mu(M) = \frac{i}{2} MM^* \in u(k) \ (\simeq u(k)^*)$.

$\mu^{-1}(\frac{i}{2} \mathrm{Id}) = \{ M / MM^* = id \}$ i.e. rows are orthonormal $\quad$ ⚠ why not 0?
$\uparrow$ central element of $\mathfrak{g}^*$ ✓

$M_{red} = \{ MM^* = id \} / U(k) \cong G(k,n)$. $\quad$ ($\dim_{\mathbb{C}} M = kn$, $\dim_{\mathbb{R}} G = k^2$, $\dim_{\mathbb{C}} M_{red} = k(n-k)$).

Ex: Polygon space: start with $SO(3) \circlearrowright S^2(r)$ sphere in $\mathbb{R}^3$, std area form, Hamiltonian.

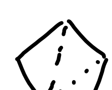
for suitable basis of $so(3)^* \leftrightarrow$ rotations about axes
$\begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & & 1 \\ & 0 & \\ -1 & & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$, $\mu : S^2 \to \mathbb{R}^3$ is the inclusion!!

Now $SO(3) \circlearrowright S^2(r_1) \times \dots \times S^2(r_n)$ $\mu(v_1, \dots v_n) = v_1 + \dots + v_n \in \mathbb{R}^3 \simeq so(3)^*$.

$$\mu^{-1}(0)/SO(3) = \{ (v_1 .. v_n) / |v_i| = r_i, \ \Sigma v_i = 0 \} / SO(3)$$
$$= \{ n\text{-gons in } \mathbb{R}^3 \text{ with given edge lengths} \} / \text{rotations \& translations.}$$

is naturally a symplectic manifold! of $\dim_{\mathbb{R}} 2n-6$.
⚠ may be singular. choose edge lengths generic to ensure $SO(3)$ action is free.
i.e. no n-gons contained in a line

Moreover, if choose a set of nonintersecting diagonals e.g.

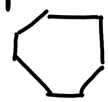


then get up to $(n-3)$ further S^1-actions – "bending actions".
(but these are only defined away from deg. polygons where diag. length = 0.

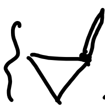
These are again Hamiltonian with μ = length of the diagonal!!

Ex. $n=4$: all equal lengths not smooth due to  but if lengths ~ 1 distinct then get $S^2 \circlearrowleft S^1$  moment map $= \ell \in [|r_1-r_2|, r_1+r_2] \cap [|r_3-r_4|, r_3+r_4]$.

$n=5$: for generic lengths close to 1, get a toric manifold  with moment map $\longleftrightarrow$ triangle ineq. for lengths

can be  $\simeq S^2 \times S^2$ blown up in 3 points.

for all lengths $=1$, still smooth and $\underset{\text{diffeo}}{\simeq} S^2 \times S^2$ blown up 3 times

but $\{$  $\} \simeq S^2$ Lag. sphere on which bending action not def'd.

but changing lengths give other spaces (eg. one length close to sum of the others $\Rightarrow \mathbb{CP}^2$)