Part 1 – Symplectic manifolds and neighborhood theorems

Possible ref. on basics: A. Cannas da Silva, "Lectures on Sympl. geometry". *Sept 6*

* **Symplectic manifolds**: (M^{2n},ω) smooth manifold, $\omega\in\Omega^2(M)$,
  - non-degenerate: $\frac{1}{n!}\omega^n\neq 0$ pointwise ($\Rightarrow$ vol. form, orientation) $\Leftrightarrow$ $v\mapsto\iota_v\omega$ $TM\xrightarrow{\sim}T^*M$.
  - closed: $d\omega=0$.

Ex:
- $M=$ oriented surface, $\omega=$ area form
- $M=\mathbb{R}^{2n}$, $\omega=\sum dx_i\wedge dy_i$
- $M=T^*N$, $\omega=d\lambda$, Liouville form $\lambda=\sum p_i\,dq_i$ in local coords. ($T^*N \xrightarrow{\pi} N$)

  intrinsically: $\lambda_{(x,\xi)}(v)=\langle\xi,d\pi(v)\rangle$.

  ω exact hence closed; clearly non-deg.
- symplectic submanifolds: $V\subset M$, $\omega_{|V}$ nondegenerate.

Q: for which n does S^{2n} have a sympl. structure? what about $T^{2n}=\mathbb{R}^{2n}/\mathbb{Z}^{2n}$?

* **Lagrangian submanifolds**: $L^n\subset(M^{2n},\omega)$, $\omega_{|L}=0$.

Ex: • zero section in T^*N, or more generally: $\alpha\in\Omega^1(N,\mathbb{R})$,

$T^*N\supset\operatorname{graph}(\alpha)$ ($\underset{\pi}{\simeq}N$ via projection). $\lambda_{|\operatorname{graph}(\alpha)}=\alpha$ (or rather $\pi^*\alpha$)

Since $\omega=d\lambda$, graph(α) is Lagrangian iff $d\alpha=0$

(exact Lag, ie. λ exact, iff α exact)

Ex: • conormal bundles: $V^k\subset N^n$ submfld $\Rightarrow$ $N^*V=\{(x,\xi)\mid x\in V,\ \xi_{|T_xV}=0\}\subset T^*N$.

rank $n-k$ bundle over V

Fact: $\lambda_{|N^*V}=0$. $\Rightarrow N^*V$ (exact) Lag.

Indeed, let $v\in T_{(x,\xi)}(N^*V)$, $d\pi(v)\in T_xV$, while $\xi\perp T_xV$, so $\lambda(v)=\langle\xi,d\pi(v)\rangle=0$.

Or ... in local coords. $(q_1,\dots,q_n)$ where $V:\{q_{k+1}=\dots=q_n=0\}$ ie. $\mathbb{R}^k_{q_1\dots q_k}$

dual coords. $(p_1,\dots,p_n)$

$\rightarrow N^*V=\left\{(q_i,p_i)\ \middle|\ \begin{array}{l}q_{k+1}=\dots=q_n=0\\ p_1=\dots=p_k=0\end{array}\right\}$ Lag. for $\omega=\sum dp_i\wedge dq_i$

ie. $\mathbb{R}^k_{q_1\dots q_k}\times\mathbb{R}^{n-k}_{p_{k+1}\dots p_n}$

Ex: graphs of symplectomorphisms: $(M,\omega)\circlearrowleft\varphi\in\operatorname{Symp}(M,\omega)$ ie. $\varphi^*\omega=\omega$, or just Diff

Then look at graph $\Gamma_\varphi=\{(x,\varphi(x))\}\subset M^-\times M$ ie. $(M\times M,\ -\pi_1^*\omega+\pi_2^*\omega)$.

aka $-\omega\oplus\omega$

Prop: $\varphi\in\operatorname{Symp}(M,\omega)\Leftrightarrow\Gamma_\varphi$ is Lagrangian.

Pf: $i:M\to\Gamma_\varphi\subset M\times M$, $x\mapsto(x,\varphi(x))$, $i^*(-\pi_1^*\omega+\pi_2^*\omega)=-(\pi_1\circ i)^*\omega+(\pi_2\circ i)^*\omega$

$=-\omega+\varphi^*\omega$.

* **Hamiltonian vector fields:**

* (M,ω) symplectic, $H \in C^\infty(M,\mathbb{R}) \rightsquigarrow \exists !\, X_H$ v.f. s.t. $\iota_{X_H}\omega = -dH$. Ham. vector field.

Recall: given time dep't v.f. (v_t): flow gen'd by v_t, ϕ_t, is family of diffeos defined by

$$\phi_0(p) = p, \quad \frac{d\phi_t(p)}{dt} = v_t(\phi_t(p))$$ (converges if M closed, or v_t has suitable growth at infinity).

general fact about flows: $\frac{d}{dt}(\phi_t^*\alpha) = \phi_t^*(\mathcal{L}_{v_t}\alpha)$

↳ Lie derivative $\left(\mathcal{L}_v\alpha = \frac{d}{dt}\Big|_{t=0} \exp(tv)^*\alpha\right)$.

Cartan's formula: $\mathcal{L}_v\alpha = d\iota_v\alpha + \iota_v d\alpha$

So: given Hamiltonians H_t, flow of X_{H_t} satisfies $\frac{d}{dt}(\phi_t^*\omega) = \phi_t^*\mathcal{L}_{X_{H_t}}\omega = \phi_t^*\, d\underbrace{\iota_{X_{H_t}}\omega}_{=-dH_t \text{ exact}} = 0$.

Conclude: $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M,\omega)$

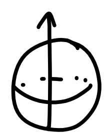
Ham diffeos (gen'd by Ham. v.f.) symplecto: $\phi^*\omega = \omega$.

· Note: $dH(X_H) = -\omega(X_H, X_H) = 0$ so flow of time indep't Ham. H preserves level sets of H.

Ex: · $\mathbb{R}^2$, $\omega_0 = dx\wedge dy = r\,dr\wedge d\theta$, $H = \frac{1}{2}r^2 \Rightarrow X_H = \frac{\partial}{\partial\theta}$ $\left(\omega(\frac{\partial}{\partial\theta}, \cdot) = -r\,dr\right)$.

uniform rotation.

· S^2, $\omega_0 = d\theta\wedge dz$



exercise: this is standard area from $\mathbb{R}^3$,

area preserving

$H = z \Rightarrow X_H = -\frac{\partial}{\partial\theta}$ uniform rotation

· T^*N, $\{$ · $H = H(q)$ factors through π $\Rightarrow X_H(q,p) = (0, -dH(q))$. flow shifts fibers (0 section ↦ graph dH)

· equip N with a Riem metric, $H = \frac{1}{2}|p|^2 \Rightarrow$ generates geodesic flow on $T^*N \underset{g}{\simeq} TN$.

• More generally, sympl. v.f. = $\iota_v\omega$ is closed (vs. Ham.: exact), flow still symplectomorphisms.

given a symplectic isotopy $(\varphi_t)_{t\in[0,1]}$ generated by sympl. v.f. v_t, define

$$\mathrm{Flux}(\varphi_t) = \int_0^1 [\iota_{v_t}\omega]\,dt \in H^1(M,\mathbb{R})$$ measures how far from Ham. isotopy.

given $\gamma: S^1 \to M$, $\varphi_t(\gamma)$ sweeps a cylinder $\Gamma(s,t) = \varphi_t(\gamma(s))$.

$$\int_\Gamma \omega = \int_0^1\int_{S^1} \omega\Big(\frac{\partial\Gamma}{\partial s}, \underbrace{\frac{\partial\Gamma}{\partial t}}_{= v_t}\Big)\, ds\, dt$$

$$= \int_0^1\int_{S^1} -\iota_{v_t}\omega\left(\frac{\partial\Gamma}{\partial s}\right) ds\, dt$$

$$= \int_0^1 \langle [-\iota_{v_t}\omega], \underbrace{[\varphi_t(\gamma)]}_{=[\gamma]} \rangle\, dt = \langle \mathrm{Flux}(\varphi_t), [\gamma] \rangle.$$

Ham. isotopy = zero area swept.

Moser's theorem

Thm (Moser): M compact closed, $(\omega_t)_{t\in[0,1]}$ smooth family of symplectic forms with $[\omega_t]\in H^2(M,\mathbb{R})$ indep of t
$\Rightarrow \exists$ isotopy $\varphi_t \in \mathrm{Diff}(M)$ st. $\varphi_t^*\omega_t = \omega_0$, in particular $(M,\omega_0) \underset{\varphi_1}{\xrightarrow{\sim}} (M,\omega_1)$.

Pf: $\frac{d\omega_t}{dt}$ is exact so $\exists\, \alpha_t$ 1-forms st. $d\alpha_t = \frac{d\omega_t}{dt}$ ← (⚠ $\exists$ smooth family α_t, using explicit formulas for Poincaré lemma w/ compact support & Mayer-Vietoris)
Let X_t st. $\iota_{X_t}\omega_t = -\alpha_t$, and φ_t flow of X_t.
Then $\frac{d}{dt}(\varphi_t^*\omega_t) \underset{\text{chain rule}}{=} \varphi_t^*\left(L_{X_t}\omega_t + \frac{d\omega_t}{dt}\right)$. But $L_{X_t}\omega_t = d\iota_{X_t}\omega_t = -d\alpha_t = -\frac{d\omega_t}{dt}$ ◀

NB:
- $\exists$ similar statements about open mflds but need extra conditions on structure at infinity. Eg: Liouville mflds = exact sympl., "convex" at infinity
- $[\omega_0]=[\omega_1]$ doesn't imply they're isotopic through ω_t in same $[\omega]$!

McDuff: $S^2\times S^2\times T^2$, $\omega_0 = \pi_1^*\omega_{S^2} + \pi_2^*\omega_{S^2} + ds\wedge dt$
$\circlearrowleft$
$\psi(z,w,s,t) = (z, R_{z,t}(w), s, t)$ $\qquad \omega_1 = \psi^*\omega_0 \qquad [\omega_1]=[\omega_0]$
↑ rotation around z angle t $\qquad \nexists\, \omega_t$ connecting them in this class (uses J-curves)

(though of course they're symplectomorphic, and deform equivt ie. $\exists\, \omega_t$ with varying $[\omega_t]$)

Thm (Darboux): $\forall p\in(M,\omega)$, $\exists$ local coords near p st. $\omega = \sum dx_i\wedge dy_i$

Lemma: $(T_pM, \omega_p) \simeq (\mathbb{R}^{2n}, \omega_0)$ as a sympl. vector space.

Pf: build standard basis $\{e_i, f_i\}$ of T_pM st. $\omega(e_i,e_j)=0 = \omega(f_i,f_j)$, $\omega(e_i,f_j)=\delta_{ij}$.
Pick e_1, then f_1 st. $\omega(e_1,f_1)=0$, then look at $\mathrm{span}(e_1,f_1)^{\perp\omega}$ (which is a complementary subspace since $\omega|_{\mathrm{span}(e_1,f_1)}$ nondeg.) and induction. ◀

Pf. Darboux: using standard basis on T_pM, $\exists$ loc. coords. ie. $\mathbb{R}^{2n}\supset U \xrightarrow{f} M$, $0\mapsto p$
st. $\omega_1 = f^*\omega$ agrees with ω_0 at the origin.
- Since nondegeneracy is an open condition, $\omega_t = (1-t)\omega_0 + t\omega_1$ is symplectic $\forall t\in[0,1]$ over a nbd. of 0 (shrink U if needed).
- Moser trick: $\exists\,\alpha$ 1-form on U st $d\alpha = \omega_1-\omega_0$, and can assume α (and its first derivatives) vanish at origin (subtract const. & linear terms!)

Then v_t st. $\iota_{v_t}\omega_t = -\alpha$, φ_t = flow of v_t, well-def! near origin
($v_t = O(\mathrm{dist}^2)$, $\varphi_t = id + O(\mathrm{dist}^2)$), flow stays in U if shrink domain further to $U'\subset U$.

Then $\frac{d}{dt}(\varphi_t^* \omega_t) = \varphi_t^*\left(L_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) = \varphi_t^*(-d\alpha + d\alpha) = 0$ so $\varphi_1^*\omega_1 = \omega_0$.

Then $(f\circ\varphi_1)^*\omega = \omega_0$. △

<u>Lagrangian nbd theorem (Weinstein):</u> Sept. 13

Recall: <u>Prop:</u> ‖ $L\subset(M,\omega)$ Lagrangian $\Rightarrow NL \simeq T^*L$.

<u>Pf:</u> $TM_{|L} \xrightarrow[\omega]{\simeq} T^*M_{|L} \xrightarrow[\text{restr.}]{} T^*L$ surjective with kernel $\supset TL$ (since $\omega_{|L}=0$) hence $= TL$ for dim. reasons

hence $NL = TM_{|L}/TL \simeq T^*L$ △

<u>Thm</u> (Weinstein). ‖ $(M,\omega)\supset L$ Lagrangian $\Rightarrow \exists$ nbds U of L in (M,ω)
U_0 of zero section L in $(T^*M, \omega_0 = d\lambda)$

and a symplectomorphism $\psi: (U_0,\omega_0) \xrightarrow{\sim} (U,\omega)$

$$\begin{array}{ccc} (U_0,\omega_0) & \xrightarrow{\sim} & (U,\omega) \\ {}_{i_0}\nwarrow & \circlearrowleft & \nearrow_{i} \\ & L & \end{array}$$

<u>Pf:</u> * pick a complement to TL, ie. subbundle $N\subset TM_{|L}$ st. $TM_{|L} = TL\oplus N$
(eg. orthogonal for some metric) - can ensure it is a Lagrangian subbundle, $\omega_{|N}=0$.
either pick $TL^\perp$ for g compatible metric ($g=\omega(\cdot,J\cdot)$, $J^2=-1$, see later).
or there's an explicit procedure for constructing a Lagrangian complement out of any complement.
(see Cannas §8.2)

* Use exp map (for any metric) to build $\psi: U_0 \subset T^*L \longrightarrow U\subset M$ s.t.
  - ψ along zero section is the inclusion $i: L \hookrightarrow M$
  - $d\psi$ along zero section maps fibers T^*L to N by the linear iso $\ell: N \xrightarrow[\omega]{\sim} T^*L$ of prop^n (or rather its inverse). eg. set $\psi(x,\xi) = \exp_x(\ell^{-1}(\xi))$

Then $\omega_1 = \psi^*\omega$ agrees pointwise with ω_0 along zero section
(pointwise: vanishes on zero section since $\omega_{|L}=0$
— " — fiber since N chosen Lagrangian as well
and pairs the two in canonical manner by choice of $d\psi$:
given $x\in L$, $\xi\in T^*_xL \subset T_{(x,0)}T^*L$, $w = \ell^{-1}(\xi)\in N_x\subset T_xM$, we have
$v\in T_xL \subset T_{(x,0)}T^*L$
$\psi^*\omega(\xi,v) = \omega(w,v) = \langle \ell(w), v\rangle = \langle \xi, v\rangle = \omega_0(\xi,v)$ ✓)

Use Moser's argument again: $\omega_t = t\omega_1 + (1-t)\omega_0$ is non-deg. (hence symplectic) on some neighborhood of the zero section (shrink U_0 if needed) and $\frac{d\omega_t}{dt} = \omega_1 - \omega_0 = d\alpha$ for some 1-form α st. $\alpha = 0$ along the zero section. (in fact to order 2).

Lemma: $U \supset L$ tubular nbd, $\beta \in \Omega^k(U)$, $d\beta = 0$, $\beta_{|L} = 0$
$\Rightarrow \exists \alpha \in \Omega^{k-1}(U)$ st. $\beta = d\alpha$ and $\alpha = 0$ at every point of L.

Moreover, if $\beta = 0$ along L then can assume α vanishes to order 2 along L.

Pf. Identify U with disc bundle in NL (here $\simeq T^*L$). Retraction $\rho_t : (x,v) \mapsto (x,tv)$

Define $\alpha_{(x,v)} = \int_0^1 \rho_t^* \left(\iota_{(0,v)} \beta \right) dt$

$\hookrightarrow v \in N_x L \subset T_{(x,tv)}(NL)$.

Then clearly vanishes along zero section (if $v = 0$, rhs $\equiv 0$)

& if $\beta_{(x,v)} = O(|v|)$ then rhs $= O(|v|^2)$.

and $d\alpha = \int_0^1 \rho_t^* \left(d\iota_{X_t} \beta \right) dt = \int_0^1 \frac{d}{dt} (\rho_t^* \beta)\, dt = \rho_1^* \beta - \rho_0^* \beta = \beta - 0$.

$\underbrace{d\iota_{X_t}\beta}_{= L_{X_t}\beta}$; $\hookrightarrow X_t(x,tv) = v$ generating ρ_t. ; ($\rho_0^*\beta = 0$: assumed)

Then let v_t st. $\iota_{v_t} \omega_t = -\alpha$ (v_t vanishes along L, in fact to order 2)

& flow φ_t (well def'd up to $t=1$ by shrinking U_0 if needed); $\frac{d}{dt}(\varphi_t^* \omega_t) = 0$.

Then $(\psi \circ \varphi_1)^* \omega = \varphi_1^* (\psi^* \omega) = \varphi_1^* \omega_1 = \omega_0 \rightarrow \varphi := \psi \circ \varphi_1 : U_0 \xrightarrow{\sim} U$. ✓

$U_0 \subset T^*L$, $U \subset M$

∎