Course overview

- Introductions
- Main goal: symplectic manifolds, Lagrangian submanifolds & Lagrangian Floer homology.

Today: overview of the course (at a rapid pace, we'll return to it more gently).

---

① * Symplectic manifolds: (M^{2n}, ω) smooth manifold, $\omega \in \Omega^2(M)$,

- non-degenerate: $\frac{1}{n!}\omega^n \neq 0$ pointwise ($\Rightarrow$ vol. form, orientation) $\Leftrightarrow$ $v \mapsto \iota_v\omega$, $TM \xrightarrow{\sim} T^*M$.
- closed: $d\omega = 0$.

Ex:
- $M =$ oriented surface, $\omega =$ area form
- $M = \mathbb{R}^{2n}$, $\omega = \sum dx_i \wedge dy_i$
- $M = T^*N$, $\omega = d\lambda$, $\lambda = p\,dq$ Liouville form
- $M = \mathbb{CP}^n$, complex proj. varieties, ...

Thm (Darboux) ‖ $\forall p \in M$ $\exists$ nbd. & local coords. (x_i, y_i) in which $\omega = \sum dx_i \wedge dy_i$

No local invariants! Obvious global invariant: $[\omega] \in H^2(M, \mathbb{R})$

Thm (Moser): ‖ M compact closed, $(\omega_t)_{t\in[0,1]}$ symplectic forms with $[\omega_t] \in H^2(M,\mathbb{R})$ indep. of t
$\Rightarrow$ $\exists$ isotopy $\varphi_t \in \mathrm{Diff}(M)$ st. $\varphi_t^*\omega_t = \omega_0$, in particular $(M, \omega_0) \xrightarrow[\varphi_1]{\sim} (M, \omega_1)$.

Moreover, the group of symplectomorphisms $\mathrm{Symp}(M,\omega)$ is very large!

$\mathrm{Symp} \supset \mathrm{Ham}(M,\omega)$ Hamiltonian diffeos = flow of (time dependent) Ham. v.f.

$H \in C^\infty(M, \mathbb{R}) \rightsquigarrow \exists!$ v.f. X_H st. $\omega(X_H, \cdot) = -dH$.

Its flow preserves ω: $L_{X_H}\omega = \underbrace{d\iota_{X_H}\omega}_{\text{exact}} + \iota_{X_H}\cancel{d\omega} = 0$.

---

② * Lagrangian submanifolds: $L^n \subset (M^{2n}, \omega)$ st. $\omega_{|L} = 0$

Ex:
- $\mathbb{R}^n_{x_i} \subset (\mathbb{R}^{2n}_{(x_i,y_i)}, \omega_0)$.
- any simple closed curve on a surface
- in T^*N, the zero section, cotangent fibers, ...
- $\prod S^1(r_i) \subset \mathbb{R}^{2n}$ or $\mathbb{CP}^n$. More generally,

  T^n-orbits in toric sympl. mflds (ie. Hamiltonian T^n-action on M^{2n})
  (= level sets of moment map $\mu = (\mu_1, \dots \mu_n)$ Ham's generating S^1-actions)

<u>Observe:</u> $TL^{\perp\omega} = TL$, so ω induces an isom. $NL = TM_{|L}/TL \xrightarrow{\sim} T^*L$

$$v \longmapsto \omega(v,\cdot)_{|TL}$$

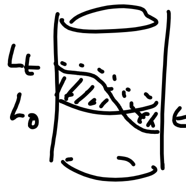
In fact, <u>Weinstein's thm:</u> ‖ a neighborhood of L in M is symplectomorphic to a nbd. of the zero section in T^*L.

Deformations of submflds $\leftrightarrow$ sections of normal bundle

In Lagr. case, a deform$^{\underline{n}}$ is Lagrangian isotopy iff $\Leftrightarrow$ graph of <u>closed</u> 1-form

Ham. isotopy $\Leftrightarrow$ <u>exact</u> 1-form.

<u>Ex:</u>



$\int \omega$ = flux of the deformation $\in \mathbb{R}$ (in general, $H^1(L,\mathbb{R})$)

measures the difference between Ham. & Lagr. isotopy.

PLAN

‖ First ~3 weeks = go over this & other classical symplectic geometry.

---

(3) <u>Classification questions:</u> what kinds of Lagr. submflds (up to Lagr. / Ham. isotopy) does M contain?

- <u>Ex:</u> on a surface, closed Lagrs $\Leftrightarrow$ s.c.c.'s, Ham. iso $\Leftrightarrow$ no area swept.
- <u>Arnold nearby Lagr. conjecture:</u> ‖ $L \subset T^*N$ closed <u>exact</u> Lagr. submanifold $\Rightarrow L$ ham. iso to the zero section.

<u>exact</u> means: $(T^*N, \omega = d\lambda)$ exact symplectic form

L Lagr. $\Leftrightarrow \lambda_{|L}$ closed

say L <u>exact</u> Lagr. if $\lambda_{|L} = df$ is exact.

e.g. graph(α) Lagr $\Leftrightarrow \alpha$ closed

exact Lagr. $\Leftrightarrow \alpha$ exact (& then Ham. iso to zero sec.).

Proved for only T^*S^1 (easy), T^*S^2 (Hind 2003), T^*T^2 (Dimitroglou Rizell, Goodman, Ivrii 2016)

though... <u>Thm</u> (Abouzaid-Kragh 2016): (after Fukaya-Seidel-Smith, Nadler, ...).

‖ $L \subset T^*N$ closed exact Lagr. $\Rightarrow \pi_{|L}: L \to N$ is a simple homotopy equivalence.

- in $\mathbb{R}^4$: $L \subset \mathbb{R}^4$ closed Lagr. $\Rightarrow \begin{cases} \text{if orientable, } L \simeq T^2 \\ \text{if not, } \chi(L) < 0 \text{ divisible by 4. (Givental)} \end{cases}$

case of Klein bottle excluded by Nemirovski 2006.

All known Lagr. tori in $\mathbb{R}^4$ are Ham. iso. to $\begin{cases} \cdot \text{ product tori } S^1(r_1) \times S^1(r_2) \\ \text{or } \cdot \text{ Chekanov torus } T_{Ch}(r). \end{cases}$

↳ ~1990 $\exists T \subset (\mathbb{R}^4, \omega_0)$ not $\simeq$ any product torus.

<u>Conj.</u> no others.

Note: Gromov: $\nexists$ closed exact Lagrangians in $\mathbb{R}^{2n}$. Next best:

- exact Lagrangian immersions: these satisfy an h-principle (Gromov)
  in particular $\exists\, L \looparrowright \mathbb{R}^{2n}$ iff. $TL \otimes \mathbb{C}$ is a trivial $\mathbb{C}$-vect-bundle.

- monotone embedded Lagrangians: given a disc $u: (D^2, \partial) \to (M, L)$,
  area $\int u^*\omega = \lambda \cdot \mu(u)$

  λ: fixed constant >0; $\mu(u)$: Maslov class = rotation # of TL around ∂D^2.

  e.g.  need equal areas

  $S^1(r) \times \ldots \times S^1(r)$ need same r

In $\mathbb{R}^6$:

- Fukaya: L closed monotone oriented Lag. $\subset \mathbb{R}^6 \iff L \simeq S^1 \times \Sigma_g$.
- Ekholm-Eliashberg-Murphy-Smith 2013: $\forall N^3$ closed oriented,
  $\exists$ exact Lag. immersion $N \looparrowright \mathbb{R}^6$ with just one double point.
  Using Lag.-surgery, $\Rightarrow \exists$ Lag. embedding $N \# (S^1 \times S^2) \hookrightarrow \mathbb{R}^6$.

* Recent developments on Lag. tori: Thm (Vianna 2014) $\|$ $\mathbb{CP}^2 \supset \infty$ly many different monotone Lag. tori.

Thm (2014) $\|$ $\mathbb{R}^6 \supset$ — " — (similarly higher dim).

(known ones all Lag. isotopic to products).

↳ whereas: Dimitroglou Rizell & Evans point out $\exists$ monotone $T^4 \subset \mathbb{R}^8$ that isn't smoothly isotopic to a product torus.

$\boxed{\text{PLAN}}$ Won't get into all these results (some of which are quite technical) but study a set of tools & explain some of the applications.

4* key tool to study Lagrangians: holomorphic discs.

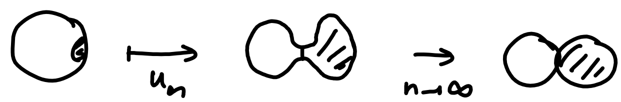
→ every sympl. mfld admits a compatible almost-cx. structure $J \in End(TM)$, $J^2 = -1$
& set of choices is contractible. $\omega(\cdot, J\cdot)$ Riem metric.

→ pseudoholomorphic curves: $u: (\Sigma, j) \to M$ s.t. $\bar\partial_J u = 0$ ie. $du \circ j = J \circ du$.
(Σ: Riem. surface)

closed, or with boundary on Lagrangians. $\mathcal{M}(J, [u], \ldots) = \{u \,/\, \bar\partial_J u = 0\}/\sim$

→ the deformation theory of J-curves is governed by a Fredholm operator ($\bar\partial$-type operator on sections of u^*TM) so expect finite-dim'l spaces of solutions, count when 0-dim!

Need to understand transversality & compactness properties of moduli spaces. Key phenomenon: bubbling & controlling it by symplectic area.
↳ local divergence that can be renormalized,



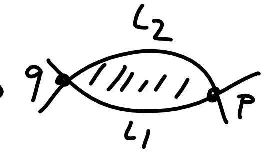
PLAN
Spend a few weeks on these foundations & then defining invariants based on this.

---

(5) • To study Lag's & their intersection properties:

Lag. Floer homology $HF(L_1,L_2) = H^*(CF(L_1,L_2),\partial)$

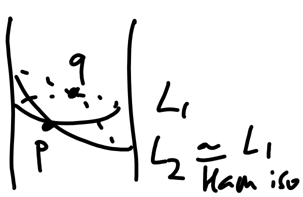
CF = vector space gen^d (over some coeff field) by $L_1 \cap L_2$ (assume $\pitchfork$)

$\langle \partial p, q\rangle$ = (weighted) count of $\{u : \mathbb{R}\times[0,1] \to$  $, \bar\partial u = 0\}/_{\sim\ \mathbb{R}\text{-transl.}}$

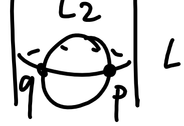
Floer: If L_i don't bound any holom. discs (eg. exact case) then
- $\partial^2 = 0$
- HF is invariant under Hamiltonian isotopies
- $HF(L,L) \simeq H^*(L)$

Corollary: then $\#(L \cap \psi(L)) \geq \dim H^*(L)$ whenever $\psi \in \mathrm{Ham}$ & $\psi(L) \pitchfork L$.

(compare: for small isotopies, $\psi(L) = \mathrm{graph}(df)$, $f \in C^\infty(L)$ Morse and the ineq. $\equiv$ Morse inequality for $\#\mathrm{crit}(f)$).

Ex: 

$\partial p = q - q = 0 \Rightarrow HF(L_1,L_2) \simeq H^*(S^1)$.

Counterex. 

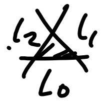
$\partial p = q,\ \partial q = p,\ \partial^2 \neq 0$.

• To study monotone Lagrangians: study counts of holomorphic discs with ∂ on L (& generally make sense of Floer theory when $\exists$ discs)
↳ obstruction in Floer theory if $\exists$ discs of Maslov index <2
↳ counts of Maslov $=2$ discs give invariants of monotone Lag's up to isotopy.
used to distinguish Chekanov tori & newer examples.

6 * Fukaya category = organize all Lag's $\subset (M,\omega)$ for which Lag.-Floer theory is well-defined & their intersection properties

obj: $L \subset (M,\omega)$

morphisms: $CF(L_1, L_2)$, $\partial = \mu^1$.

composition: $\mu^2 : CF(L_1,L_2) \otimes CF(L_0,L_1) \longrightarrow CF(L_0,L_2)$ Floer product counts holom. 

+ higher operations μ^k

Point: → easier to understand Lag's up to Floer-theoretic iso. than up to Ham. iso.
↳ ie. how intersects all other Lag's.

→ thanks to homological algebra, it may be enough to understand Floer theory for a finite set of generators of $\mathcal{F}(M)$, expressing all others in terms of these. For instance, when $M = T^*N$,

→ compact exact Fukaya cat. is generated by zero section

→ wrapped Fukaya cat. (noncompact exact Lag's with specific perturbations at infinity) is generated by cotangent fiber.

* Besides symplectic geometry, Fukaya categories also play a key role in

~> mirror symmetry: Fukaya cat of M ⟷ derived cat. of coherent sheaves on $M^\vee$

~> low-dim. top: Fukaya cat. of certain configuration spaces are where various invts of 3-mflds, knots & links live naturally.
Eg. Heegaard-Floer homology = Floer homology in $\mathrm{Sym}^g(\Sigma_g)$ but also Khovanov homology (Seidel-Smith / Abouzaid).