

Math 241 Homework 3, due 3/8/11

1. (a) On $\mathbb{P}^1$, let D be the divisor which takes value 1 at ∞ and 0 everywhere else. For $n \in \mathbb{Z}$, determine the global sections of $\mathcal{O}_{nD}$ and Ω_{nD}^1 , then verify the Riemann-Roch formula for $\mathcal{O}_{nD}$ by determining its cohomology groups.

(b) Let L_n be the holomorphic line bundle corresponding to the divisor nD . Describe L_n in terms of trivializations and transition functions. What is its degree? Show that every holomorphic line bundle over $\mathbb{P}^1$ is isomorphic to L_n for some $n \in \mathbb{Z}$.

2. Let τ be a complex number with $\text{Im } \tau > 0$, and let $X = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

(a) Let $\phi(z) = \exp(-2\pi iz - \pi i\tau)$, and let $c \in \mathbb{C}$. Show that the quotient L_c of $\mathbb{C} \times \mathbb{C}$ by the equivalence relation $\sim$ generated by $(z, v) \sim (z + 1, v)$ and $(z, v) \sim (z + \tau, \phi(z + c)v)$ is a degree 1 holomorphic line bundle over X .

(b) Define the *theta function*

$$\vartheta(\tau, z) = \sum_{m \in \mathbb{Z}} \exp\left(2\pi i\left(\frac{1}{2}m^2\tau + mz\right)\right).$$

Show that $z \mapsto \vartheta(\tau, z + c)$ defines a holomorphic section of L_c . What is the space of global sections $H^0(X, \mathcal{O}_{L_c})$?

(c) Show that the holomorphic line bundles L_c and $L_{c'}$ are isomorphic if and only if $c - c' \in \mathbb{Z} + \tau\mathbb{Z}$.

(d) Interpret the result of (c) in terms of the exact sequence

$$\cdots \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}^*) \rightarrow \cdots$$

and the classification of degree 1 holomorphic line bundles over X .

3. (Exercise 17.5 in Forster.) Let K be a canonical divisor on a compact Riemann surface X of genus $g > 0$, and let $D \geq K$ be a divisor with $\deg D = \deg K + 1$. Show that the sheaf $\mathcal{O}_K$ is globally generated, but $\mathcal{O}_D$ is not.

(Hint: write $D = K + p$ where $p \in X$, and compare $H^0(X, \mathcal{O}_K)$ with $H^0(X, \mathcal{O}_D)$.)