MIRROR SYMMETRY: LECTURE 7

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1. Degenerations and Monodromy (contd.)

Last time, we considered families $\mathcal{X} \xrightarrow{\pi} D^2$ where for $t \neq 0, X_t \cong X$ (with varying J) and for $t = 0$, X_0 is typically singular. We saw that monodromy around $t = 0$ induces $\phi_* \in \mathrm{Aut}(H^n(X_{t_0}, \mathbb{Z}))$.

Theorem 1. *All eigenvalues of ϕ_* are roots of unity: thus $\exists N, k$ s.t. $(\phi_*^N - \mathrm{id}\,)^k = 0$. Moreover, $k \leq n + 1$.*

Replacing ϕ by ϕ^N (the "base change" $X'_t = X_{t^N}$), we can assume that ϕ_* is unipotent, i.e. $(\phi_* - \mathrm{id})^k = 0$. It is maximally unipotent if $k = n + 1$. We can further define a weight filtration associated to a unipotent ϕ_* coming from the Jordan block decomposition of ϕ_*: letting

$$N = \log(\phi_*) = (\phi_* - \mathrm{id}) - \frac{(\phi_* - \mathrm{id})^2}{2} + \cdots + (-1)^{n+1} \frac{(\phi_* - \mathrm{id})^n}{n} \tag{1}$$

act on $V = H^n(X, \mathbb{Q})$, we obtain a filtration $0 \subseteq W_0 \subseteq \cdots \subseteq W_{2n} = V$ s.t. $N(W_i) \subset W_{i-2}$ and $N^k : W_{n+k}/W_{n+k-1} \xrightarrow{\sim} W_{n-k}/W_{n-k-1}$. We construct this as follows:

- First, $N^n : W_{2n}/W_{2n-1} \xrightarrow{\sim} W_0$ so $W_0 = \mathrm{im}\,(N^n), W_{2n-1} = \mathrm{Ker}\,(N^n)$.
- Then let $V' = W_{2n-1}/W_0$, so N induces $N' \in \mathrm{End}(V')$ (since $W_{2n-1} = \mathrm{Ker}\,N^n \supseteq \mathrm{im}\,N$ and $W_0 = \mathrm{im}\,(N^n) \subseteq \mathrm{Ker}\,N$) with $(N')^n = 0$. By induction, we obtain

$$0 \subseteq W'_0 \cong W_1/W_0 \subseteq \cdots \subseteq W'_{2n-2} \cong W_{2n-1}/W_0 = V' \tag{2}$$

and

$$W_{2n-2} = \{v \mid N^{n-1}(v) \in W_0 = \mathrm{im}\,N^n\} \supseteq \mathrm{im}\,N \tag{3}$$

so $W_{2n} \xrightarrow{N} W_{2n-2}$. Finally, $W_1 = \{N^{n-1}(v) \mid N^n(v) = 0\} \subset \mathrm{Ker}\,N$ so $W_1 \xrightarrow{N} 0$, and we obtain $W_k \to W_{k-2}$ by induction.

Example. For the elliptic curves from last time, with $\phi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, we have $0 \subseteq W_0 \subseteq W_1 \subseteq W_2 = H^1(C, \mathbb{Q}) \cong \mathbb{Q}^2$, with $W_0 = W_1 = \mathrm{im}\,N = \mathrm{Ker}\,N = \mathrm{Span}(a)$ being the direction invariant by monodromy.

Note that if N is the $(n+1)\times(n+1)$ Jordan block with 0's on the diagonal and 1s above (with columns e_i), then $W_0 = \mathrm{Span}(e_1), W_{2n-1} = \mathrm{Span}(e_1 \cdots e_n)$, and we can reduce to the equivalent $(n-1)\times(n-1)$ Jordan block and repead the process with $W_1 = W_0, W_{2n-2} = W_{2n-1}, \cdots, W_{2k-2} = W_{2k-1} = \mathrm{Span}(e_1 \cdots e_k)$. There is a similar story if N is a sum of such Jordan blocks.

Remark. In fact, the interplay of weight filtration with Hodge filtration

$$F^p = H^{n,0} \oplus \cdots \oplus H^{p,n-p} \qquad (H^n = F^0 \supseteq F^1 \supseteq \cdots, F^p/F^{p+1} \cong H^{p,n-p}) \tag{4}$$

(with Griffiths transversality giving $\nabla F^p \subseteq F^{p-1}$ under deformations) gives a notion of "mixed Hodge structure". By [Schmid], there exists a limiting Hodge filtration as $t \to 0$, but we won't say any more about those.

Now consider a multidimensional family $\mathcal{X} \to (D^2)^s$ smooth over $(D^*)^S$ where $D^* = D^2 \smallsetminus \{0\}$. Then we have s monodromies $\phi_1, \ldots, \phi_s \in \mathrm{Aut} H_n(X)$, $[\phi_i, \phi_j] = 0$ (since $\pi_1((D^*)^s) = \mathbb{Z}^s$ is abelian), so $N_i = \log \phi_i$ also commute.

Theorem 2 (Cattani-Kaplan)**.** *All the elements of the form $\sum \lambda_i N_i, \lambda_i > 0$ have the same monodromy weight filtration.*

We want to consider a "universal family" of Calabi-Yau manifolds near a "deepest corner", caled a "large complex structure limit point" in the moduli space.

Definition 1 (Morrison)**.** *Given a family of Calabi-Yau n-folds $\mathcal{X} \to (D^*)^S \subset (D^2)^s$, $s = h^{n-1,1}(X)$, s.t. the Kodaira-Spencer map $T_*(D^*)^s \to H^1(TX_t)$ is an isomorphism at every point of $(D^*)^s$, we say that $0 \in (D^2)^s$ is a* large complex structure limit (LCSL) point *if*

(1) *The monodromies ϕ_j around each factor are all unipotent.*
(2) *Let $N_j = \log \phi_j, N = \sum \lambda_j N_j$ for $\lambda_j > 0$ arbitrary. Then the weight filtration $0 \subseteq W_0 \subseteq W_1 \subseteq \cdots \subseteq W_{2n} = H^n(X,\mathbb{Q})$ has* $\dim W_0 = \dim W_1 = 1, \dim W_2 = \dim W_3 = s+1$.
(3) *Let α_0^* be the generator of W_0, $\alpha_1^*, \cdots, \alpha_s^*$ the rest of a basis for W_2. Then $\exists m_{jk} \in \mathbb{Q}$ s.t. $N_j(\alpha_k^*) = m_{jk}\alpha_0^*$, i.e. $\phi_j(\alpha_k^*) = \alpha_k^* + m_{jk}\alpha_0^*$. We further require that (m_{jk}) is an invertible matrix.*

This essentially says that the family is locally a "full deformation", that we single out a one-dimensional subspace $\mathrm{Span}(\alpha_0^\vee)$ of $H^n(X)$ preserved by the monodromy, and that, for each factor D^2, we get a class $\tilde{\alpha}_j^*$ s.t. $\phi_j(\tilde{\alpha}_j^*) = \tilde{\alpha}_j^* + \alpha_0^*$ and $\tilde{\alpha}_j^*$ is invariant under the other ϕ_i.

Remark. If $h^{n-1,1} = s = 1$, then this is equivalent to the statement that the monodromy around zero is maximally unipotent. For instance, the family of elliptic curves seen last time is an LCSL point.

Now, for a family of Calabi-Yau 3-folds, we have by definition

$$0 \subset \underbrace{W_0 = W_1}_{\dim\ =1} \subset \underbrace{W_2 = W_3}_{\dim\ =s+1=h^{2,1}+1} \subset \underbrace{W_4 = W_5}_{\dim\ =2s+1} \subset \underbrace{W_6 = H^3(X;\mathbb{Q})}_{\dim\ =2s+2} \tag{5}$$

where we use $N^k : W_{n+k}/W_{n+k-1} \xrightarrow{\sim} W_{n-k}/W_{n-k-1}$ to get the dimensions of W_3, W_4, W_5. Now, $H^3(X)$ carries an intersection pairing preserved by ϕ_*, so $N = \log \phi_*$ is in the Lie algebra, i.e. $(x, Ny) + (Nx, y) = 0$.

Lemma 1. $W_{4-2i} = W_{2i}^{\perp}$.

Proof. Since $W_0 = \text{im } N^3, W_4 = W_5 = \text{Ker } N^3$, $(x, N^3y) = -(N^3x, y) = 0$ for $x \in W_4, N^3y \in W_0$ and the dimensions match. Furthermore, $N(W_4) = W_2$ (it is onto since $N : W_4/W_3 \xrightarrow{\sim} W_2/W_1$ and $W_0 = \text{im } N^3 = N(\text{im } N^2)$): thus, for $x, Ny \in W_2$, $(x, Ny) = -(Nx, y) = 0$ (since $W_0 \perp W_4$) and the dimensions match. □

Finally, passing to $H_3(X, \mathbb{Q})$ by Poincaré duality, let $S_i = PD(W_i)$ (or equivalently, viewing $H_3 = (H^3)^*$, S_i is the annihilator of W_{4-2i}).

Proposition 1. *Given an LCSL point in the moduli space of Calabi-Yau 3 folds with $h^{2,1} = s$, $\exists$ a $\mathbb{Z}$-basis $(\alpha_0, \dots, \alpha_S, \beta_0, \dots, \beta_S)$ of $H_3(X, \mathbb{Z})$ s.t. $\beta_0 \in S_0$, $\beta_1, \dots, \beta_s \in S_2$, $\alpha_1, \dots, \alpha_s \in S_4, \alpha_0 \in S_6 = H_3(X)$ s.t. $(\alpha_i, \alpha_j) = (\beta_i, \beta_j) = 0, (\alpha_i, \beta_j) = \delta_{ij}$.*

Proof. Let β_0 be the $\mathbb{Z}$ generator of S_0 (unique up to sign), which we extend to a $\mathbb{Z}$-basis β_i of S_2. By the lemma, S_2 is Lagrangian w.r.t. the intersection product, so $(\beta_i, \beta_j) = 0$. Let β_i^* be the dual basis of $S_2^* = H^3/W_2$, i.e. $\beta_i^* \beta_j = \delta_{ij}$, and let $\alpha_i \in H_3$ be the Poincaré dual of some lift of β_i^* to H^3. Then $(\alpha_i, \beta_j) = \delta_{ij}$. We can make $(\alpha_i, \alpha_j) = 0$ inductively by replacing α_i with $\alpha_i - \sum(\alpha_i, \alpha_j)\beta_j$. Finally, $\alpha_1, \dots, \alpha_s \in S_4$ since $(\alpha_i, \beta_0) = 0$ and $S_4 = S_0^{\perp}$. □

We now define canonical coordinates on our moduli space. Given $\mathcal{X} \to (D^*)^s$ LCSL, let $\Omega(t_1, \dots, t_s)$ be the holomorphic volume form on $X_{(t_1,\dots,t_s)}$, normalized so that $\int_{\beta_0} \Omega(t_1, \dots, t_s) = 1$. Set $w_i(t_1, \dots, t_s) = \int_{\beta_i} \Omega(t_1, \dots, t_s)$. This is not quite a coordinate because of monodromy: as t_j goes around the origin, $\beta_i \mapsto \phi_j(\beta_i) = \beta_i - m_{ji}\beta_0$ for some $m_{ji} \in \mathbb{Z}$ (an integer since these are integer classes). In fact, these are the m_{ji} from the definition of LCSL. Instead, we set $q_i = \exp(2\pi i w_i)$: these are well-defined functions on $(D^*)^s$, and are canonical once the basis $\{\beta_i\}$ is chosen. Note that q_i is a zero of order $-m_{ji}$ (i.e. a pole of order m_{ji}) along $t_j = 0$; if the m_{ji}'s are nonpositive, then we get coordinates on $(D^2)^s$, and can choose a basis of S_2 appropriately.

Example. For our elliptic curves from last time, $q = \exp(2\pi i \tau(t)), \tau(t) = \int_b \Omega$ where $\int_a \Omega = 1$.

If e_i is a basis of $H^2(\check{X},\mathbb{Z}), e_i$ in the Kähler cone, we obtain coordinates on the complexified Kähler moduli space: if $[B+i\omega] = \sum \check{t}_i e_i$, let $\check{q}_i = \exp(2\pi i \check{t}_i), \check{t}_i = \int_{e_i^*} B + i\omega$.

Example. In example above, we have $\check{q} = \exp(2\pi i \int_{T^2} B + i\omega)$.

Conjecture 1 (Mirror Symmetry)**.** *Let $f : \mathcal{X} \to (D^*)^S$ be a family of Calabi-Yau 3-folds with LCSL at 0. Then $\exists$ a Calabi-Yau 3-fold $\check{X}$ and choices of bases $\alpha_0, \dots, \alpha_S, \beta_0, \dots, \beta_S$ of $H_3(X,\mathbb{Z})$, $e_1, \dots, e_S$ of $H^2(X,\mathbb{Z})$ s.t. under the map $m : (D^*)^S \to \mathcal{M}_{Kah}(\check{X})$, $(q_1, \dots, q_S) \mapsto (\check{q}_i, \dots, \check{q}_S)$, $\check{q}_i = q_i$, we have a coincidence of Yukawa couplings*

$$\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \rangle_p^X = \langle \frac{\partial}{\partial \check{q}_i}, \frac{\partial}{\partial \check{q}_j}, \frac{\partial}{\partial \check{q}_k} \rangle_{m(p)}^{\check{X}} \tag{6}$$

where the LHS corresponds to $\int_X \Omega \wedge (\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}\frac{\partial}{\partial q_k}\Omega)$ and the RHS to a $(1,1)$-coupling, i.e. the Gromov-Witten invariants $\langle e_i, e_j, e_k \rangle_{0,\beta}^{\check{X}}$ (since $2\pi i \check{q}_i \frac{\partial}{\partial \check{q}_i} = \frac{\partial}{\partial \check{t}_i} = e_i \in H^{1,1}$).