MIRROR SYMMETRY: LECTURE 5

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1. Gromov-Witten Invariants

Recall that if (X,ω) is a symplectic manifold, J an almost-complex structure, $\beta \in H_2(X,\mathbb{Z})$, $\overline{\mathcal{M}}_{g,k}(X,J,\beta)$ is the set of (possibly nodal) J-holomorphic maps to X of genus g representing class β with k marked points up to equivalence. This is not a nice moduli space, but does have a fundamental class $[\overline{M}_{g,k}(X,J,\beta)] \in H_{2d}(\overline{M}_{g,k}(X,J,\beta),\mathbb{Q})$, where $2d = \langle c_1(TX),\beta\rangle + 2(n-3)(1-g)+2k$. We further have an evaluation map $\mathrm{ev} = (\mathrm{ev}_1,\dots,\mathrm{ev}_n) : \overline{M}_{g,k}(X,J,\beta) \to X^k, (\Sigma, z_1,\dots,z_k,u) \mapsto (u(z_1),\dots,u(z_k))$. Then the Gromov-Witten invariants are defined for $\alpha_1,\dots,\alpha_k \in H^*(X), \sum \deg \alpha_i = 2d$ by

$$\langle \alpha_1,\dots,\alpha_k\rangle_{g,\beta} = \int_{[\overline{M}_{g,k}(X,J,\beta)]} \mathrm{ev}_1^*\alpha_1 \wedge \cdots \wedge \mathrm{ev}_k^*\alpha_k \in \mathbb{Q} \tag{1}$$

Or dually, for $\alpha_i = PD(C_i)$, $\#(\mathrm{ev}_*[\overline{M}_{g,k}(X,J,\beta)] \cap (C_1 \times \cdots \times C_k)) \in \mathbb{Q}$.

For a Calabi-Yau 3-fold, we're interested in $g = 0, k = 3$, so $\Sigma = (S^2, \{0,1,\infty\})$. For $\deg \alpha_i = 2, \alpha_i = PD(C_i)$, C_i cycles transverse to the evaluation map, we have

$$\begin{aligned}\langle \alpha_1,\alpha_2,\alpha_3\rangle_{0,\beta} = \#\{&u : S^2 \to X \ J\text{-hol. of class } \beta,\\ &u(0) \in C_1, u(1) \in C_2, u(\infty) \in C_3\}/\sim\end{aligned} \tag{2}$$

Reparameterization acts transitively on triples of points, so

$$\begin{aligned}\langle \alpha_1,\alpha_2,\alpha_3\rangle_{0,\beta} &= (C_1\cdot\beta)(C_2\cdot\beta)(C_3\cdot\beta)\#\{u : S^2 \to X \ J\text{-hol. of class } \beta\}/\sim\\ &= (\int_\beta \alpha_1)(\int_\beta \alpha_2)(\int_\beta \alpha_3)\cdot \#[\overline{\mathcal{M}}_{0,0}(X,J,\beta)]\end{aligned} \tag{3}$$

We denote by $N_\beta \in \mathbb{Q}$ the latter number $\#[\overline{\mathcal{M}}_{0,0}(X,J,\beta)]$. This works when $\beta \neq 0$: when $\beta = 0$, we instead obtain

$$\langle \alpha_1,\alpha_2,\alpha_3\rangle_{0,0} = \int_X \alpha_1\wedge\alpha_2\wedge\alpha_3 \tag{4}$$

1.1. **Yukawa coupling.** Physicists write this as

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{0 \neq \beta \in H_2(X,\mathbb{Z})} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} e^{2\pi i \int_\beta B + i\omega} \tag{5}$$

We want to ignore issues of convergence, and so treat this is a formal power series

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{\beta \neq 0} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} q^\beta \in \Lambda \tag{6}$$

where Λ is the completion of the group ring $\mathbb{Q}[H_2(X,\mathbb{Z})] = \{\sum a_i q^{\beta_i} | a_i \in \mathbb{Q}, \beta_i \in H_2\}$. Specifically, we allow infinite sums provided that $\int_{\beta_i} \omega \to +\infty$.

1.2. **Quantum cohomology.** This is new product structure on $H^*(X)$ deformed by this coupling. Namely, pick a basis (η_i) of $H^*(X)$, (η^i) the dual basis, i.e. $\int_X \eta_i \wedge \eta^j = \delta_{ij}$. Set

$$a_1 * a_2 = \sum_i \langle \alpha_1, \alpha_2, \eta^i \rangle \eta_i = \alpha_1 \wedge \alpha_2 + \sum_{\beta \neq 0} \langle \alpha_1, \alpha_2, \eta^i \rangle_{0,\beta} q^\beta \eta_i \tag{7}$$

Definition 1. *The* quantum cohomology *of X is $QH^*(X) = (H^*(X;\Lambda), *)$.*

Theorem 1. *This is an associative algebra.*

The proof of this relies on understanding the relationship between 4 point GW invariants and various 3 point ones.

1.3. **Kähler moduli.** We can view q as the coordinates on a Kähler moduli space: for (X,J)-complex, the Kähler cone $\mathcal{K}(X,J) = \{[\omega] | \omega \text{ Kahler}\} \subset H^{1,1}(X) \cap H^2(X,\mathbb{R})$ is a open, convex cone. Its real dimension is $h^{1,1}(X)$, and we can make it a complex manifold by adding the "B-field".

Definition 2. *Let (X,J) be a Calabi-Yau 3-fold with $h^{1,0} = 0$ (so $h^{2,0} = 0$ and $H^{1,1} = H^2$). Then the* complexified Kähler moduli space *is*

$$\begin{aligned} \mathcal{M}_{Kah} &= (H^2(X,\mathbb{R}) + i\mathcal{K}(X,J))/H^2(X,\mathbb{Z}) \\ &= \{[B + i\omega], \omega \ \textit{Kahler}\}/H^2(X,\mathbb{Z}) \end{aligned} \tag{8}$$

Choose a basis (e_i) of $H^2(X,\mathbb{Z})$, $e_1, \ldots, e_m \in \overline{\mathcal{K}(X,J)}$ (which exists by openness). We can write $[B + i\omega] = \sum t_i e_i, t_i \in \mathbb{C}/\mathbb{Z}$, so we have coordinates on $\mathcal{M}_{Kah}$ given by $q_i = \exp(2\pi i t_i)$. Thus, $\mathcal{M}_{Kah}$ is an open subset of $(\mathbb{C}^*)^m$ which contains $(\mathbb{D}^*)^m$, where $\mathbb{D}^* = \{q | 0 < |q| < 1\}$.

We now can associate q^β to $q_1^{d_1} \cdots q_m^{d_m}$, where $d_i = \int_\beta e_i$ for $e_i \geq 0$ integers (it is an integer cohomology class integrated against an integer homology class): explicitly, $q_1^{d_1} \cdots q_m^{d_m} = \exp(2\pi i \sum d_i t_i) = \exp(2\pi i \int_\beta B + i\omega)$. We can view $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ as a power series in the q_i, though we still do not know about convergence.

1.4. **Gromov-Witten invariants vs. numbers of curves.** We have, for $\alpha_1, \alpha_2, \alpha_3 \in H^2(X)$,

$$(9)\quad \begin{aligned} \langle \alpha_1, \alpha_2, \alpha_3 \rangle &= \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{\beta \neq 0} \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,\beta} q^\beta \\ &= \int_X \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \sum_{\beta \neq 0} (\int_\beta \alpha_1)(\int_\beta \alpha_2)(\int_\beta \alpha_3) N_\beta q^\beta \end{aligned}$$

This is much like our formula from the first class, except the latter term had the form $n_\beta \frac{q^\beta}{1-q^\beta}$ and n_β as the number of "rational curves of class β". The discrepancy comes from the existence of multiple covers. Let $C \subset X$ be an embedded rational curve in a Calabi-Yau 3-fold. A theorem of Grothendieck says that a holomorphic bundle over $\mathbb{P}^1$ splits as $\bigoplus \mathcal{O}_{\mathbb{P}^1}(d_i)$, where $\mathcal{O}(d)$ is the sheaf whose sections are homogeneous degree d holomorphic functions on $\mathbb{C}^2$ and $\mathcal{O}(-1)$ is the tautological bundle. Writing $NC \cong \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2)$, we obtain

$$(10)\quad 0 = c_1(TX)[C] = c_1(NC)[C] + c_1(TC)[C] = d_1 + d_2 + 2$$

so $d_1 + d_2 = -2$. The "generic case" is $d_1 = d_2 = -1$, in which case C is automatically regular as a J-holomorphic curve. The contribution of C to the Gromov-Witten invariant $N_{[C]}$ is precisely 1. On the other hand, there is a component $\mathcal{M}(kC) \subset \mathcal{M}_{0,0}(X, J, k[C])$ consisting of k-fold covers of C. What is $\#[\mathcal{M}(kC)]$?

Theorem 2. *If $NC \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then the contribution of C to $N_{k[C]}$ is $\frac{1}{k^3}$.*

There are various proofs, all of which are somewhat difficult. For instance, Voisin shows that $\exists$ perturbed $\overline{\partial}$-equations $\overline{\partial}_J u = \nu(z, u(z))$ s.t. the moduli space $\tilde{MM}_3(kC)$ (of perturbed J-holomorphic maps with 3 marked points representing $k[C]$ and whose image lies in a neighborhood of C) is smooth and has real dimension 6. Moreover, $(\mathrm{ev}_1 \times \mathrm{ev}_2 \times \mathrm{ev}_3)_*[\tilde{\mathcal{M}}_3(kC)] = [C \times C \times C] \in H_6(X \times X \times X)$. Then the contribution of C to $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,k[C]}$ is

$$(11)\quad \int_{ev_*[\tilde{\mathcal{M}}_3]} \alpha_1 \times \alpha_2 \times \alpha_3 = (\int_C \alpha_1)(\int_C \alpha_2)(\int_C \alpha_3) = \frac{1}{k^3}(\int_{kC} \alpha_1)(\int_{kC} \alpha_2)(\int_{kC} \alpha_3)$$

We expect that (*) $N_\beta = \sum_{\beta = k\gamma} \frac{1}{k^3} n_\gamma$.

Remark. We do not know if n_γ is what we think it is, but we use this formula as a definition; see the Gopakumar-Vafa conjecture, which claims that $n_\gamma \in \mathbb{Z}$, and the theory of Donaldson-Thomas invariants and MNOP conjectures.

Assuming (*), we have

$$\sum_\beta (\int_\beta \alpha_1)(\int_\beta \alpha_2)(\int_\beta \alpha_3) N_\beta q^\beta = \sum_{k,\gamma} (\int_{k\gamma} \alpha_1)(\int_{k\gamma} \alpha_2)(\int_{k\gamma} \alpha_3) \frac{n_\gamma}{k^3} q^{k\gamma} \tag{12}$$
$$= \sum_\gamma (\int_\gamma \alpha_1)(\int_\gamma \alpha_2)(\int_\gamma \alpha_3) n_\gamma \sum_{k \geq 1} k^{k\gamma}$$

Where we are headed: we correspond this pairing to

$$\langle \theta_1, \theta_2, \theta_3 \rangle = \int_X \Omega \wedge (\nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega) \tag{13}$$

on $H^{2,1}(\check{X})$.