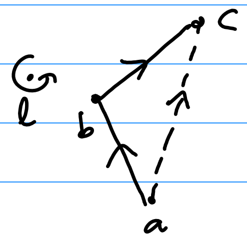
18.937 Lecture 6 — Wed Mar 1

Markov's thm: describes how the various closed braid representations of a same link isotopy type are related to each other
& as a consequence, gives an alg. criterion for two elements of B_n to represent the same link.

Thm: (Markov 1935)

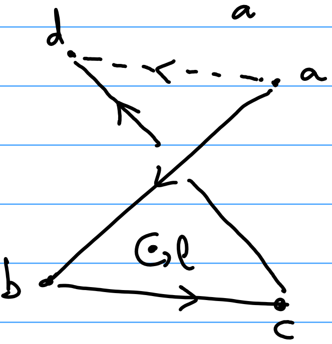
V, V' two closed braids which represent isotopic links $\Rightarrow$ $\exists$ finite sequence of closed braids $V = V_0, V_1, \dots, V_k = V'$ s.t. V_i & V_{i-1} differ by an operation which is either



$\mathcal{E}^b_{ac}$ involving only positive edges ("$\mathcal{E}^+$")

in both cases, the triangle abc / the tetrahedron abcd don't intersect any other parts of the link

or

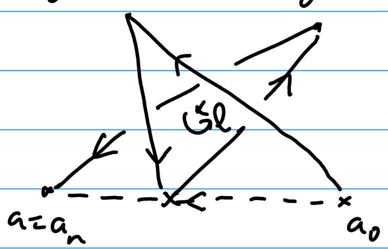


W^{bc}_{ad} (involves only positive edges; changes # strings by one)

Note: we can always pass from V to V' by moves of type $\mathcal{E}$ but not necess. preserving property of being a closed braid (i.e. all edges > 0).

Goal: given a sequence of moves $V \rightsquigarrow V'$ (not through closed braids), modify it so we have only positive edges at each stage, & do $\mathcal{E}^+$ and W moves only.

Recall: given a link, can get rid of a neg. edge by adding a sawtooth. ("$\mathcal{J}$" move).



Goal: do the same in 1-param families (along the moves $V \rightsquigarrow V'$)

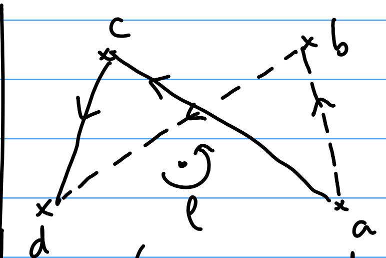
$\Rightarrow$ if we pass from a link to another one by an $\mathcal{E}$ operation, can we deduce a way to relate the corresponding closed braids (obtained by adding sawteeth) via $\mathcal{E}^+$ & W?

tons of cases to consider depending on whether edges $>0, <0$

Lemma 1: Can always assume all links appearing in the sequence of moves are in general position (i.e. have no edges coplanar with ℓ)

(clear: move pts just a tiny bit)

Lemma 2:

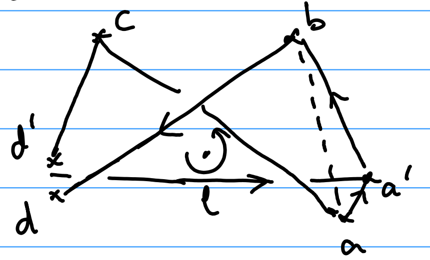


can move a pt keeping edges >0 by a sequence of $\mathcal{E}^+$ & W moves.

(assumptions: acd, abd triangles don't intersect the rest of the link)

Pf: • If $(a,d]$ positive then obvious ($(\mathcal{E}^+)^{-1}$ to remove b, then add c)

• If $[a,d]$ negative:



1) $\mathcal{E}^+$ to insert a'
2) W to insert c and d' b/w a, a'
3) W^{-1} remove a' & b
4) $(\mathcal{E}^+)^{-1}$ remove d'.

Lemma 3: Assume V has a negative edge $[a,a']$ on which sawteeth can be erected in 2 different manners: the resulting $\mathcal{S}^{b_1 \ldots b_n}_{a_0 \ldots a_n}(V)$ & $\mathcal{S}^{d_1 \ldots d_m}_{c_0 \ldots c_m}(V)$ are related by a sequence of $\mathcal{E}^+$ & W moves. (subdivision of $[a,a']$)

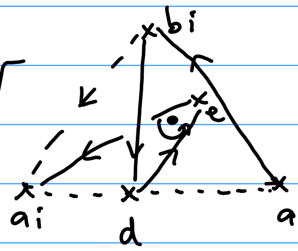
($\Rightarrow$ all possible ways of turning a given link into a closed braid are equivalent up to $\mathcal{E}^+, W$)

Pf: • can always refine a sawtooth to a finer subdivision of $[a,a']$:

e.g. inserting an extra point $d \in (a, a_i)$; by operations W (& $\mathcal{E}^+$);

If $b_i d$ positive:

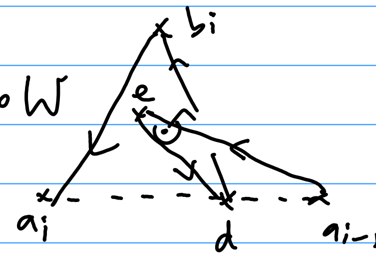
do W



pick e in right region, very close to $\Delta(a_{i-1}\, a_i\, b_i)$ (so tetrahedron $\not\in$ link)

If $b_i d$ negative:

do W



similarly,

(if $b_i d$ passes through ℓ, first move b_i a little bit by Lemma 2)

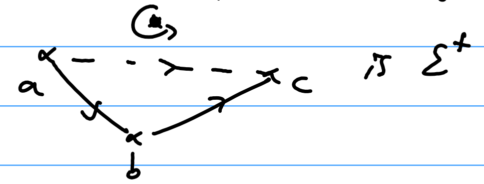
• so: given 2 sawteeth $\mathcal{S}, \mathcal{S}'$ first refine $\mathcal{S}$ to a common subdivision of the base segment, then use lemma 2 to move each tooth to the proper location, then undo refinement to get $\mathcal{S}'$. ∎

Then, left with: given an $\mathcal{E}$ operation $V_1 \rightsquigarrow V_2$ the corresponding closed braids $\bar{V}_1, \bar{V}_2$ obtained by adding sawteeth are related by $\mathcal{E}^+$ & $\mathcal{W}$ moves. (ie: can pass from sawteeth on neg edges of V_1 to sawteeth on neg. of V_2)

→ study all 8 cases of $\mathcal{E}_{ac}^{b}$ depending on signs of ab, bc, ac

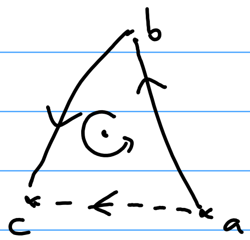
By lemma 3, just need to be able to transform ac (if >0) or our favorite sawtooth on ac (if <0) into same for ab+bc. (don't touch any sawteeth we've put on other neg edges)

∘ 2 cases already taken care of:



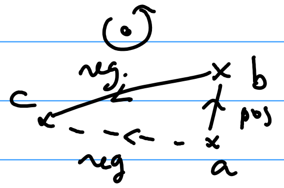
is $\mathcal{E}^+$

and



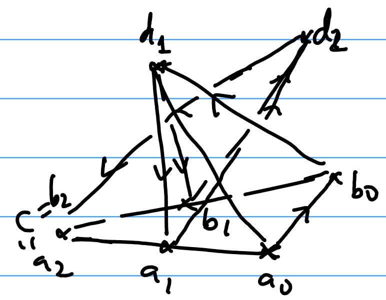
is $\mathcal{S}$ so sawtooth on V_1 (using b for edge ac) ≡ sawtooth on V_2!

• e.g. if



- if a & b very close to each other, then ∃ common sawtooth

ie. pts $a_0 = a, a_1, \dots a_n = c$
$b_0 = b, b_1, \dots b_n = c$
$d_1 \dots d_n$ s.t.



• $\mathcal{S}^{d_1 \dots d_n}_{a_0 \dots a_n}$ & $\mathcal{S}^{d_1 \dots d_n}_{b_0 \dots b_n}$ are valid sawteeth

• each pyramid w/ vertices $a_i, a_{i+1}, b_i, b_{i+1}, d_{i+1}$ doesn't meet the considered links (including edges of sawteeth added on other edges) in any unwanted places

• $(a_i b_i)$ not coplanar with ℓ (just choose a_i, b_i generic on segments) (just take a sawtooth on ac & "move" it over to bc).

Then: $a_0 d_1 a_1 d_2 \cdots a_{n-1} d_n c$ (closed braid corresp to initial links)

$\downarrow$ $\longleftarrow$ if $a_{n-1} b_{n-1} > 0$: $\mathcal{E}^{b_{n-1}}_{a_{n-1} d_n}$

replace one a_i by b_i successively starting from the right

$a_0 d_1 \cdots a_{n-2} d_{n-1} b_{n-1} d_n c$

get $\cdots d_{n-1} a_{n-1} b_{n-1} d_n \cdots$

then $\left(\mathcal{E}^{a_{n-1}}_{d_{n-1} b_{n-1}}\right)^{-1}$

$\downarrow$

$- - - -$

$\downarrow$

$a_0 d_1 b_1 d_2 \cdots b_{n-1} d_n c$

else similarly $\mathcal{E}^{b_{n-1}}_{d_{n-1} a_{n-1}}$

$\downarrow \mathcal{E}^{b_0}_{a_0 d_1}$

get $\cdots d_{n-1} b_{n-1} a_{n-1} d_n \cdots$

$a_0 b_0 d_1 b_1 \cdots b_{n-1} d_n c$

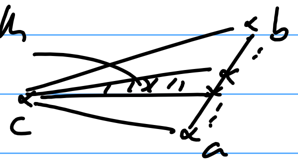
then $\left(\mathcal{E}^{a_{n-1}}_{b_{n-1} d_n}\right)^{-1}$.

(the final links + sawtooth)

(valid because of our assumption on pyramids. Can't necess use the construction of Lemma 2) since it assumes $\nexists$ triangles avoid V)

In general: subdivide [ab] into small subintervals s.t.

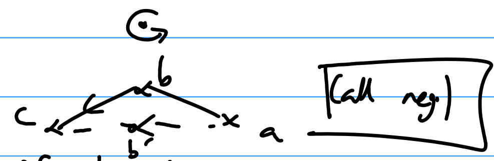
$\exists$ common sawtooth



Compactness of [a,b] $\Rightarrow$ get them in finitely many steps

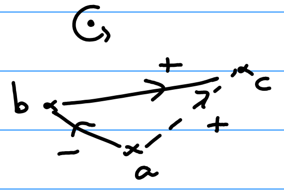
- Similarly in other cases:

E.g.:

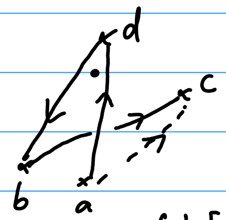


$\leadsto$ if b close to $b' \in [a,c]$, find common sawteeth for ab & ab', and for bc & b'c

otherwise bring b to b' in small steps as above

- 

If a very close to b then $\exists$ tooth adb

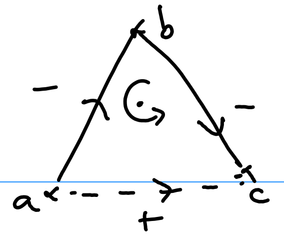
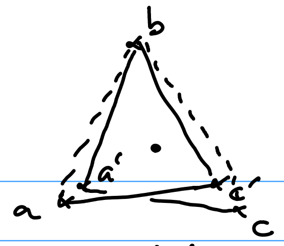


& this is W

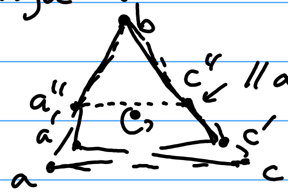
this & $\int$ case complete when 1 of the 3 is negative

If not, subdivide ab into small steps

$\leadsto$ end up with sawtooth on ab + bc by sequence of W ✓.

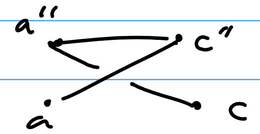
•  $\Rightarrow$ 

is a sawtooth on a-b-c staying very close to triangle abc (& hence avoiding rest of V)

(ε^+) twice to get  $c'' \parallel ac$ very close above l.

two + above = complete when 2 of the 3 are negative

then $(\varepsilon^+)^{-1}$ to remove a', c', b from picture

get  $= W$ on $a' \overset{+}{-} c$

$\Longrightarrow$ all cases covered

∎

Corollary:

$\hat\beta$ closed braid corresp. to $\beta \in B_n$

$\hat\beta'$ ——— $\beta' \in B_{n'}$

$\hat\beta$ & $\hat\beta'$ represent same oriented link isotopy type iff can modify $\beta = \beta_1 \rightsquigarrow \beta_2 \rightsquigarrow \cdots \rightsquigarrow \beta_s = \beta'$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \cap \qquad\quad \cap \qquad\qquad\quad \cap$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad B_{n_1=n} \quad B_{n_2} \;\; \cdots \;\; B_{n_s=n'}$

s.t. each move is either (M_I): $\underset{n_i}{\beta_i} \longmapsto \underset{n_i}{b\beta_i b^{-1}}$, $b \in B_n$ (conjugation)

or (M_{II}) $\underset{n_i}{\beta_i} \longmapsto \underset{n_i+1}{\beta_i \sigma_{n_i}^{\pm 1}}$ (stabilization)

or vice versa (destabilization)

• Clearly M_I, M_{II} don't modify the link isotopy type $\rightarrow$ show pictures.