lecture 22 — May 8

We've seen: X proj. surface, $f : X \to \mathbb{CP}^2$ generic projection

$\rightsquigarrow$ $\begin{cases} D \text{ branch curve} = \text{simple Hurwitz curve w/ cusps \& nodes} \\ \qquad (\xleftrightarrow{1-1} \text{fact}^{\text{ns}} \text{ of } \Delta^2 \text{ up to Hurwitz + conj. equivalence}) \\ \theta : \pi_1(\mathbb{CP}^2 - D) \to \mathfrak{S}_n \text{ satisfying some conditions} \end{cases}$

$\to$ the data (fact^n, θ) is a purely algebraic description of the top. of the covering...

Q: what if we start with any (fact^n, θ) satisfying the alg. conditions

$\to$ ie. D simple Hurwitz, $\theta : \pi_1(\mathbb{CP}^2 - D) \to \mathfrak{S}_n$. What do we get?

A: a symplectic branched covering.

Symplectic geom: recall (X^{2n}, ω) is symplectic if $\omega \in \Omega^2(X)$, $d\omega = 0$, $\underbrace{\omega \wedge \dots \wedge \omega}_{n} > 0$

- Darboux's thm.: near any point, $\exists$ local coords. in which $\omega = dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n$.
- Moser's stability thm.: $(\omega_t)_{t \in [0,1]}$ sympl. forms on X compact mfld, $[\omega_t] \in H^2(X, \mathbb{R})$ constant $\Rightarrow \exists (\phi_t)_{t \in [0,1]} : X \to X$ s.t. $\phi_t^* \omega_t = \omega_0$.

call this a "sympl. isotopy"

Ex: • a Kähler form on a $\mathbb{C}$ mfld is symplectic

so, e.g., $\omega_{FS} = i\partial\bar\partial \log |z|^2$ on $\mathbb{CP}^n \to$ restrict to $X \subset \mathbb{P}^n$ alg. gives a sympl. structure.

• the symplectic category is much richer. (first examples: Thurston, 70s)

E.g.: Hodge theory $\Rightarrow$ a Kähler mfld has b_1 even

But Gompf 1994: any finitely presented G is $\pi_1(X^4, \omega)$.

• So: not every symplectic mfld has a $\mathbb{C}$ structure; but $\exists$ compatible almost-$\mathbb{C}$ structure, ie. $J \in \text{End}(TX)$, $J^2 = -Id$, $\omega(\cdot, J\cdot)$ is a Riem. metric.

(In fact, space of compatible acs's is contractible).

The difference: no constraint on ∇J ("integrability")

$[T^{1,0}, T^{1,0}] \not\subset T^{1,0}$; $\bar\partial^2 \neq 0$; $\nabla J \neq 0$

(obstruction: Nijenhuis tensor)

Rmk: among 4-mflds, COMPLEX SURFS $\underset{\neq}{\subset}$ SYMPL. 4-MFLDS $\underset{\neq}{\subset}$ SMOOTH 4-MFLDS

surgery constructions, e.g. Gompf — cohomological obstruction; Seiberg-Witten theory

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Def: X^4 oriented; (Y^4, ω_Y) symplectic. We say $f: X \to Y$ is a <u>symplectic branched covering</u> if $\forall p \in X$, $\exists$ local coordinates near p - <u>oriented</u> - and $f(p)$ - <u>adapted</u> [see below] in which f is one of the 3 models $\mathbb{C}^2 \to \mathbb{C}^2$

$$(x,y) \mapsto (x,y)$$
$$(x,y) \mapsto (x^2,y)$$
$$(x,y) \mapsto (x^3 - xy, y)$$

- <u>Adapted coordinates</u> := $\varphi: U \subset (Y, \omega_Y) \to \mathbb{C}^2$ local diffeo s.t. $(\varphi_* \omega_Y)$ (the sympl form viewed in the coordinates) is positive on complex lines, ie. $\varphi_* \omega_Y (v, iv) > 0$ $\forall v \neq 0$.

<u>Equivalently</u>: any complex curve $\subset \mathbb{C}^2$ is a <u>sympl. submfld</u> of Y, ie. <u>restriction of ω is an area form.</u>

<u>Rmk</u>: So... the branch curve $D \subset Y$ is a sympl. submfld of Y (with immersed sings & ordinary cusps); $R \subset X$ smooth submfld.

<u>Prop</u>: $f: X \to Y$ sympl. branched covering $\Rightarrow \exists$ sympl. form ω on X, with $[\omega] = f^*[\omega_Y]$; in fact, $\exists$ <u>canonical</u> ω up to symplectic isotopy

<u>Pf</u>: $f^*\omega_Y$ is closed, nondegenerate outside of R, but degenerate (in direction of $\ker df$) along R.

<u>Claim</u>: $\exists \alpha$ <u>exact</u> 2-form s.t. $\alpha_{|\ker df} > 0$ at every point of R (naturally oriented by local model)

- Then, take $\omega = f^*\omega_Y + \varepsilon\alpha$ for $\varepsilon > 0$ small enough:

ω is closed, and $\omega \wedge \omega = \underbrace{f^*(\omega_Y \wedge \omega_Y)}_{\substack{\geq 0 \text{ everywhere} \\ > 0 \text{ outside of } R}} + 2\varepsilon \underbrace{f^*\omega_Y \wedge \alpha}_{\substack{>0 \text{ along } R \\ \text{by choice of } \alpha}} + \varepsilon^2 \underbrace{\alpha \wedge \alpha}_{\text{any}}$

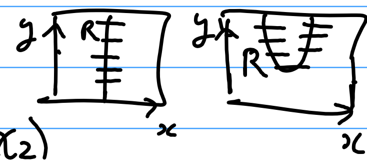
$\Rightarrow \omega \wedge \omega > 0$ everywhere if ε small enough.

- Moreover, $\{\alpha \,/\, \alpha \text{ exact}, \alpha_{|\ker df} > 0\}$ is <u>convex</u> so can interpolate b/w any two ω's; exactness $\Rightarrow [\omega_t]$ is constant; Moser's thm $\Rightarrow$ sympl. isotopy.

(Pf claim: calc. in local model. In both models, $\ker df = \langle x\text{-axis}\rangle$.
Take $\alpha = d(\chi_1(x)\,\chi_2(y)\,u\,dv)$

$u = \mathrm{Re}\,x$, $v = \mathrm{Im}\,x$

cut-off functions for a "box"



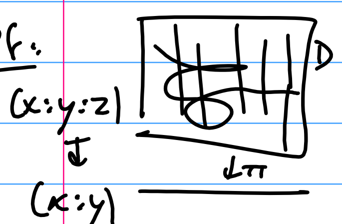
so $\chi_1 \equiv 1$ along $R \cap \mathrm{supp}(\chi_1\chi_2)$

& sum these over open cover of R). ▲

Next, observation: Any simple Hurwitz curve $D \subset \mathbb{CP}^2$ can be isotoped among simple Hurwitz curves to a symplectic submfld.

NB: Complex curves are symplectic ($TD = \langle v, iv\rangle \Rightarrow \omega_{|TD} > 0$ as $\omega(v, iv) > 0$)
but being sympl. is much easier - at each pt, just want TD closer to complex than to anti-complex - can deviate by "up to 90°" at each pt ...

Pf:



recall $\mathbb{CP}^2 - pt$ = total space of $\mathcal{O}(1)$
$\downarrow$
$\mathbb{CP}^1$

Rescale fiber directions:
$(x:y:z) \mapsto (x:y:\lambda z)$ gives D_λ.

For $\lambda \to 0$, D_λ shrinks to a nbd of the zero section
and CV is C^1 outside of a nbd of vertical tangencies,
ie. TD_λ converges to 0-section as well.

Now: - away from tangencies, $\omega_{|TD_\lambda} > 0$ because the zero section is a sympl. submfld.
- near tangencies, ok by local model. ▲

In fact, this constr. is canonical up to isotopy among sympl. submflds:
if D, D' sympl. & Hurwitz, isotopic as Hurwitz curves
$\Rightarrow$ scale down the family to get an isotopy among sympl. H. curves

Moreover, our branched covers w/ sympl. H. branch curves satisfy assumptions of the prop$^{\underline{n}}$. ⟶

Corollary: To every ($\text{Fact}^{\underline{n}}$, θ) (satisfying the alg. cond.), can associate a sympl. 4-mfld (X, ω) and a sympl. covering $f: X \to \mathbb{CP}^2$; these are canonically determined up to isotopy.

So... our missing branched covers correspond to sympl. 4-mflds !!

Q: how many sympl. 4-mflds can we get in this way?

Observe: (up to choice of normalization factor), standard $\omega_{\mathbb{CP}^2}$ has the property that $[\omega_{\mathbb{CP}^2}] =$ generator of lattice $H^2(\mathbb{CP}^2, \mathbb{Z}) \subset H^2(\mathbb{CP}^2, \mathbb{R})$

So for a sympl. covering of $\mathbb{CP}^2$ we must have:

$[\omega] = [f^*\omega_{\mathbb{CP}^2}]$ is always an integer cohomology class.

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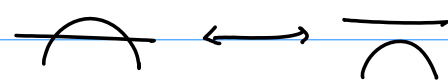
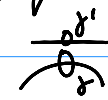
does not restrict diffeo type of X: deform ω by a small closed 2 form so $[\omega]$ rat'l, then take a multiple

Thm. (X^4, ω) compact sympl. mfld, $[\omega]$ integer class

$\Rightarrow$ for all large enough integer k, $\exists$ sympl. branched covering $f_k : X \to \mathbb{CP}^2$, with the following properties:

- the sympl. form on X induced by f_k & $\omega_{\mathbb{CP}^2}$ is isotopic to $k\omega$
- the branch curve D_k is a simple H.-curve w/ cusps, nodes, and <u>neg nodes</u> as sings.
- for suff. large k, $\exists$ canonical way of constructing f_k, up to isotopies & node cancellations



<u>NB:</u>
- unknown whether statement can be improved so that neg. nodes don't occur at all ???
- node creations / cancellations must be performed compatibly w/ θ ie. can only create a pair of nodes if  $\theta(\gamma), \theta(\gamma')$ are disjoint transpositions. (else X becomes singular)

In fact,

Thm (A.-Kulikov-Sharchishin).

D_1, D_2 simple Hurwitz curves w/ $\pm$nodes & cusps, irreducible. If $\deg D_1 = \deg D_2$, same #cusps, same (# +nodes − # negnodes) then D_1, D_2 are equivalent up to $\begin{cases} - \text{isotopy of H. curves} \\ - \text{creation / cancellation of nodes.} \end{cases}$

Csq: Let $\mathcal{D}$ = set of pairs $\left\{\begin{array}{l}\bullet \text{ a factoriz}^n \text{ of } \Delta^2 \text{ into (halftwists)}^{1,\pm 2,3} \\ \bullet \text{ a compatible } \theta: \pi_1(\mathbb{CP}^2 - D) \to S_n\end{array}\right\}$,

up to natural equivalence relations
(conjug^n, Hurwitz equiv., node creation/cancellation)
compatibly w/ θ

$\mathcal{S} = \{(X^4, \omega),\ \omega$ integral sympl. form$\}$ / symplectomorphism

Then we have natural maps $\mathcal{D} \longrightarrow \mathcal{S}$

$\mathcal{S} \longrightarrow$ sequences (for $k \gg 0$) of elts of $\mathcal{D}$

& composition one way is $(X,\omega) \leadsto (\text{fact}^n_k, \theta_k) \leadsto (X, k\omega)$
(but the other way could be anything a priori?? don't know if all sympl. coverings of $\mathbb{CP}^2$ are obtained by the method of the Thm. - actually, believe NOT).

Idea of thm: given (X,ω), choose
- J compatible a.c.s.
- $\begin{array}{c} L \\ \downarrow \\ X\end{array}$ $\mathbb{C}$ line bundle, $c_1(L) = [\omega]$
- ∇ connection on L w/ curvature $-2\pi i\omega$
  $\to \partial, \bar\partial$-operators on sections of L:
  $\bar\partial s = \frac{1}{2}(\nabla s + i\nabla s \cdot J)$

If (X,ω,J) Kähler then L is an ample holom. line bundle
$\Rightarrow$ for $k \gg 0$, $L^{\otimes k}$ has many holom. sections (define an embedding $X \hookrightarrow \mathbb{CP}^N$)
so choosing 3 sections generically gives $f: x \mapsto (s_0(x) : s_1(x) : s_2(x))$ branched covering ✓.

If J is only an a.c.s. then $\bar\partial^2 \neq 0$ and in fact $\nexists$ holom. sections of $L^{\otimes k}$
($\nexists$ local holom. fns in general!)

Still, Donaldson's observation: for $k \gg 0$, $L^{\otimes k}$ has "approx. hol" sections, ie. s.t.
$\sup |\bar\partial s| < \frac{C}{\sqrt{k}} \sup |\partial s|$; find 3 sections with good enough transversality properties so that $(s_0 : s_1 : s_2)$ is a branched covering w/ the desired properties

In particular, want:
- (s_0, s_1, s_2) don't vanish simultaneously
- $\det \partial f = s_0\, \partial s_1 \wedge \partial s_2 + \dots$ vanishes transversely along a smooth sympl. submfld R
- $\partial f_{|R}$ vanishes transversely ... & much more!

"approx. hol. transversality theory"