Lecture 21 — Wed Apr 3

- We've seen: $X \subset$ proj. surf $\Rightarrow$ a generic projection $f: X \xrightarrow[n:1]{} \mathbb{CP}^2$ is a branched cover, whose branch curve D has cusp & node singularities
- Study plane curves w/ cusps & nodes in more detail — show: can assume D is a simple Hurwitz curve ($\Rightarrow$ characterized by its b.m.f.).
- **Prop:** For generic choice of linear proj. $\pi: \mathbb{CP}^2 - pt \to \mathbb{CP}^1$, D is a simple Hurwitz curve (i.e. only special pts of D wrt π are cusps, nodes, & nondeg. vert. tangents; & all in $\neq$ fibers of π).

Pf: - D has finitely many flexes, i.e. pts where tgt has contact order ≥ 3.
Indeed, flexes of $D \Leftrightarrow$ pts where D osculates its tangent line to order 3 ($\Leftrightarrow$ "curvature $= 0$")
$=$ alg subvar. of D.
Either this alg subvar. is a finite set, or it contains a component of D – which is then a linear $\mathbb{CP}^1$.
In any case, choosing the pole of π – not on any line $\subset D$
– not on tangent at any of the flexes
ensures that wherever D is tangent to a fiber of π, the tangency is nondegenerate.

- Also avoid all lines tangent to a branch of D at a node
lines tangent to D at a cusp ($\Rightarrow$ correct local models at cusps & nodes)
- Finally, avoid: lines through 2 singular pts of D
lines through a sing. pt. of D & tangent to D elsewhere
(these are finitely many) $\Rightarrow$ special pts lie in distinct fibers ∎

$\Rightarrow D$ is described by its braid monodromy factorization
$[\text{halftwists}, (\text{halftwists})^2, (\text{halftwists})^3]$ up to H.equiv. + conjug.

- Thm ("Zariski-Van Kampen")

If $D \subset \mathbb{CP}^2$ is a simple Hurwitz curve of degree d, then $\pi_1(\mathbb{CP}^2 - D)$ has a presentation with generators $\gamma_1, \dots, \gamma_d$ & relations
- $\gamma_1 \cdots \gamma_d = 1$
- for each factor $\beta^{-1}\sigma_1\beta$ in the bmf, $\underset{\uparrow}{\beta_*}(\gamma_1) = \beta_*(\gamma_2)$
braid group action on free gp.

• ______ $\beta^{-1}\sigma_1^2\beta$ ______ , $\beta_\alpha(\gamma_1\gamma_2)=\beta_\alpha(\gamma_2\gamma_1)$

• ______ $\beta^{-1}\sigma_1^3\beta$ ______ , $\beta_\alpha(\gamma_1\gamma_2\gamma_1)=\beta_\alpha(\gamma_2\gamma_1\gamma_2)$

(and so on if A_n singularities)

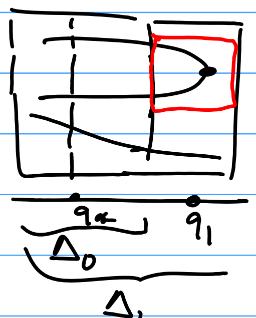
NB: analogy with: $\pi_1(S^3-K)$, & presentation when K is a closed braid (D = "2-knot")

Sketch pf: first consider $\pi_1(\mathbb{C}^2-D)$; let $q_\star$ = base pt in $\pi_1(\mathbb{C}-\{q_i\})$ and $\ell=\pi^{-1}(q_\star)\simeq\mathbb{C}$; $\pi_1(\ell-(\ell\cap D))=\langle\gamma_1,\dots,\gamma_d\rangle$ $\quad$ proj^{ns} of special pts

(By def= this is the free gp on which the factors of the braid monodromy act).

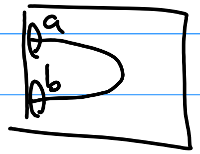
start with a disc $\Delta_0\ni q_\star$ containing no other q_i, so $(\Delta_0\times\mathbb{C})-D\underset{h.e.}{\sim}\ell-D$.

Enlarge Δ_0 successively to include more and more of the q_i's, adding them one at a time:

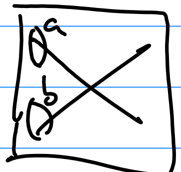


$(\Delta_1\times\mathbb{C})-D$ retracts onto the union of $(\Delta_0\times\mathbb{C})-D$ w/ a small nbd of the special pt above q_1. Apply Van Kampen's thm to this union...

• Easy to check: $\pi_1(\mathbb{C}^2-\{y^2=x^{n+1}\})=\langle a,b \mid \underbrace{ab\dots}_{n+1}=\underbrace{ba\dots}_{n+1}\rangle$



a=b

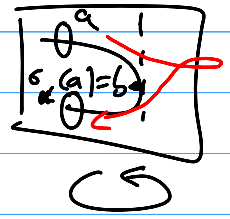


ab=ba

- -

(in fact: this relation says a = its image under the braid monodromy:

(sliding a around the special pt)



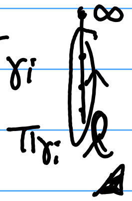
σ_1: $a\mapsto b$
$\quad b\mapsto b^{-1}ab$

so $(\sigma_1)_\star a=b$ $\qquad A_0$: $a=b$
$(\sigma_1^2)_\star a=b^{-1}ab$ $\qquad A_1$: $a=b^{-1}ab$
$(\sigma_1^3)_\star a=b^{-1}a^{-1}bab$ $\qquad$ -- --
---- . $\qquad$ -- --

- Attaching map identifies
  a & b with conjugates of generators adapted to the halfturn being performed — namely, $\beta_*(\gamma_1)$ and $\beta_*(\gamma_2)$ ...

so Van Kampen says $\pi_1((\Delta_1\times\mathbb{C})-D) = \pi_1((\Delta_0\times\mathbb{C})-D) / \langle 1 \text{ relation between } \beta_*(\gamma_1) \,\&\, \beta_*(\gamma_2)\rangle$.

- Repeat process to get a presentation of $\pi_1(\mathbb{C}^2-D)$.
  Finally, glue in the line at infinity in $\mathbb{CP}^2$
  $\Longleftrightarrow$ (by Van Kampen) quotient π_1 by meridian to line at ∞, i.e. by $\prod \gamma_i$



Note: $\pi_1(\mathbb{CP}^2-D)$ is an isotopy invariant of the curve $D\subset\mathbb{CP}^2$

So ... one could try to use it as an invariant to study the $\mathbb{C}$ projective surface X. Analogy with: π_1 of knot complement determines the knot. $>10^4$ pages written on calculation of π_1's for various examples of branch curves. Strategy: ① compute braid monodromy of D, ② apply Van Kampen & simplify presentation to something manageable. Remarkably this works on many examples (but it's very technical & computational ...)

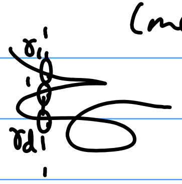
Unfortunately, things don't seem headed in the right direction, (this is my PERSONAL opinion!) & I believe this program will fail to produce new invariants — one should be looking at the bmf, not at π_1 (which only sees the subgroup of B_d generated by the factors). Of course, how to compare bmf's up to Hurwitz equivalence is unknown ...

To study a branched cover $f: X\to\mathbb{CP}^2$

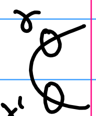
- The other ingredient is $\theta: \pi_1(\mathbb{CP}^2-D)\to S_n$ monodromy of the branched cover. It has some properties related to local structure of covering f above the special pts:

Recall (Zariski-Van Kampen)

$\pi_1(\mathbb{CP}^2 - D)$ has generators $\gamma_1 \dots \gamma_d$ (meridians around D)

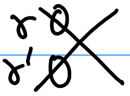


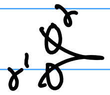
- at a tangency, equality relation b/w 2 conjugates of γ_i



$\gamma = \gamma'$ (if b. monodromy $\beta^{-1}\sigma_1\beta$, $\gamma = \beta_*(\gamma_1)$, $\gamma' = \beta_*(\gamma_2)$)

- at a node, $\gamma\gamma' = \gamma'\gamma$



- at a cusp  $\gamma\gamma'\gamma = \gamma'\gamma\gamma'$

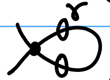
- θ must map each γ_i to a transposition

- θ is onto (provided X is connected) ($\text{Im}(\theta)$ = subgp of $\mathfrak{S}_n$ gen^d by transpositions & acting transitively on $\{1..n\}$)

- $\theta(\gamma) = \theta(\gamma')$ (automatically holds)

- $\theta(\gamma), \theta(\gamma')$ disjoint transpositions (ie. w/out a common index) because branching occurs in different sheets!!

- $\theta(\gamma), \theta(\gamma')$ adjacent (one common index).

- We can recover the $\mathbb{C}$ surface X from the branch curve D & the morphism θ. In fact, given D there's only finitely many possibilities at most for θ (since $\pi_1(\mathbb{CP}^2 - D)$ finitely gen^d & $\mathfrak{S}_n$ finite); each of the θ's satisfying the above conditions determines a proj. surface X_θ & a covering $X_\theta \to \mathbb{CP}^2$. Of course, sometimes $\nexists\, \theta$ satisfying the conditions (e.g. if  $\theta(\gamma), \theta(\gamma')$ equal or disjoint??)

Chisini conjecture: (essentially proved by Kulikov 1998) || Given any alg plane curve D w/ cusps & nodes, if $\exists\, \theta : \pi_1(\mathbb{CP}^2 - D) \to \mathfrak{S}_n$, $n \geq 5$ satisfying the above conditions, then θ and n are unique (up to conjugation by an elt of $\mathfrak{S}_n$).

(Counterexample: $\mathbb{CP}^2 \hookrightarrow \mathbb{CP}^5 \xrightarrow[4:1]{} \mathbb{CP}^2$ generic deg 2 polynomial map

Branch curve has deg. 6 & 9 cusps – we'll see it later)

$\exists$ 3 morphisms $\pi_1(\mathbb{CP}^2 - D) \to \mathfrak{S}_4$ satisfying the conditions
1 morphism $\mathfrak{S}_3$

This is believed to be the only alg. example where D doesn't determine X_θ; Kulikov +... reduces potential list to finitely many (small) cases.

Examples: ① $\mathbb{CP}^1 \times \mathbb{CP}^1 \simeq$ quadric $\{uv = zw\} \subset \mathbb{CP}^3$

(embedding by $\mathcal{O}(1,1)$: products of homog. coords. on the 2 factors).

change coords $(4ab = (a+b)^2 - (a-b)^2)$

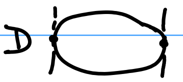
→ quadric

$$\begin{tikzcd}
\{z_0^2 + z_1^2 = z_2^2 + z_3^2\} \arrow[r, phantom, "\subset"] \arrow[dr, "2:1"'] & \mathbb{CP}^3 \arrow[d] & (z_0 : \dots : z_3) \arrow[d, mapsto] \\
 & \mathbb{CP}^2 & (z_0 : z_1 : z_2)
\end{tikzcd}$$

branched at curve $D : \{z_0^2 + z_1^2 = z_2^2\}$ ⟶ smooth conic

(this corresponds to double roots for z_3)

(affine part: $\left(\frac{z_0}{z_2}\right)^2 + \left(\frac{z_1}{z_2}\right)^2 = 1$)



bmf $\Delta^2 = \sigma_1 \cdot \sigma_1$

$\theta : \pi_1(\mathbb{CP}^2 - D) = \mathbb{Z}/2 \xrightarrow{\sim} S_2 = \mathbb{Z}/2$

$(\langle \gamma_1, \gamma_2 \mid \gamma_1\gamma_2 = 1, \gamma_1 = \gamma_2 \rangle)$ $(\gamma_1, \gamma_2 \mapsto (1\,2))$.

②

$$\begin{tikzcd}
\mathbb{CP}^2 \arrow[r, hook, "i", "\text{all deg. 2 polynomials}"'] \arrow[rr, bend right, "f"'] & \mathbb{CP}^5 \arrow[r, "p", "\text{generic}"'] & \mathbb{CP}^2
\end{tikzcd}$$

generic quadratic map = 4:1 covering

R has degree 3

D has degree 6

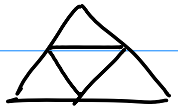
- 9 cusps
- no nodes
- 3 vert tangencies

Can compute in 2 different ways:

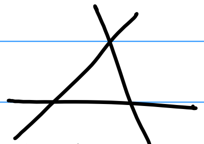
- degenerate the image of i (a deg. 4 surface) into a union of 4 planes intersecting along 3 lines (use e.g. toric geometry)

Moishezon Teicher

approach to many other examples!

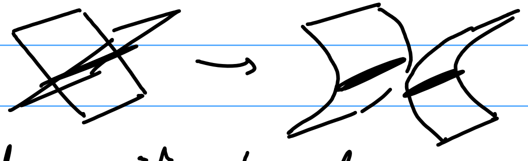


Then for $X_0 \xrightarrow[p]{} \mathbb{CP}^2$, get



3 lines (p(intersection))

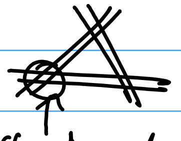
but ... when "regenerate" (smooth X_0 back to $X = i(\mathbb{CP}^2)$)



along an intersection line

each intersection line doubles up.

get

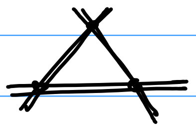


analyze: stuff at each of these pts.!

- or: degenerate f to an easier map (non-generic),

$(x : y : z) \mapsto (x^2 : y^2 : z^2)$

R = union of 3 coord-lines



$D = f(R) =$ ______ with multiplicity 2 !!

Perturbation needed, esp near the 3 pts $(0:0:1)$, $(0:1:0)$, $(1:0:0)$.
Local model same near each of them: start w/

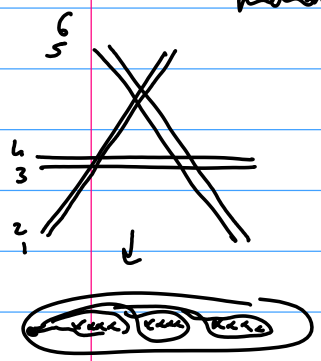
$$(x,y) \longmapsto (x^2, y^2)$$

replace by $\quad (x,y) \longmapsto (x^2+\varepsilon_1 y,\ y^2+\varepsilon_2 x)$

Study the branch curve (image of $\{\det df = 0\} = \{4xy = \varepsilon_1\varepsilon_2\}$)

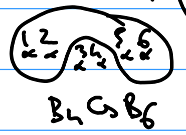
& a generic proj$^{\underline{n}}$ of it ($\boxed{\triangle}$ not to coord. axes! take $(z_1,z_2) \mapsto z_1 + \varepsilon' z_2$).

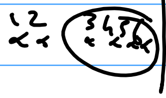
$\Rightarrow$ get 3 cusps + 1 vert. tangency at each of the 3 pts.

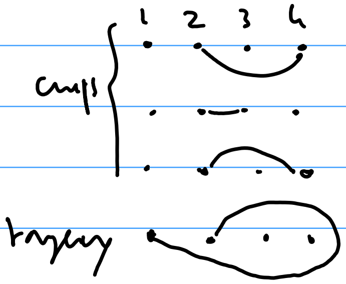


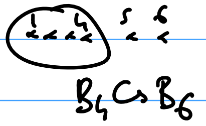
monodromy fact$^{\underline{n}}$ at the end: e.g. ($\exists$ many Hurwitz equiv. expressions)

$$\left(\underline{\sigma_{24}}^3 \cdot \sigma_{23}^3 \cdot \overline{\sigma_{24}}^3 \cdot \widetilde{\sigma_{12}}\right) \cdot \left(\text{same for}\right)\left(\text{same for}\right)$$









Exercise: check product is Δ_6^2 ✓.