Lecture 2 – Monday Feb 13 – 18.937

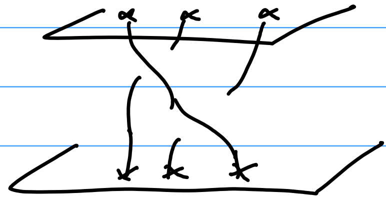
Def: M manifold (think: of dim 2)

$\widetilde{\mathcal{C}}_n(M) = \{(z_1, .., z_n) \in M^n / z_i \neq z_j \; \forall i \neq j\}$ ordered configurations.

$\mathcal{C}_n(M) = \widetilde{\mathcal{C}}_n(M) / \mathfrak{S}_n$ unordered

$\pi_1(\mathcal{C}_n(M)) := B_n(M)$ <u>braid group of M</u> (motion of n unordered pts in M)

In particular, $B_n := B_n(\mathbb{R}^2)$. <u>braid group</u> (Artin 1925)



- Representation of a homotopy class = "<u>geometric braid</u>" $a \subset M \times [0,1]$
  = graph of $f(t) = (f_1(t), .., f_n(t))$, $f_i : [0,1] \to M$
  $f_i(t)$ distinct $\forall t$, $f_i(0)$ & $f_i(1)$ coincide up to permutation.

- Def^n ⇒ 2 geom braids represent the same elt of $B_n(M)$ iff $\exists$ homotopies $h_i : [0,1]^2 \to M$ s.t. $a(s) = \{h_i(s,\cdot)\}$ geom braid $\forall s \in [0,1]$, $a(0) = a$, $a(1) = a'$
  [say: a, a' are <u>equivalent</u>]

Prop (Artin 1947) ← for $\mathbb{R}^2$

2 geom. braids a, a' repres the same elt of $B_n(M)$ iff $\exists$ isotopy $(\varphi_s)_{s \in [0,1]}$,
$\varphi_s \in \mathrm{Homeo}(M \times [0,1])$, $\varphi_0 = \mathrm{Id}$, $\varphi_{s\,|M \times \{0\}} = \mathrm{Id}$, $\varphi_{s\,|M \times \{1\}} = \mathrm{Id}$,
$\varphi_s(a) = a'$, and
(*) $\forall s \in [0,1]$, $\varphi_s(a)$ is a geometric braid (ie. $\forall t$, $\varphi_s(a) \cap (M \times \{t\})$ = n distinct pts)

Pf:
- if $\exists$ isotopy φ_s as in prop then $a_s = \varphi_s(a)$ gives the homotopy b/w a & a' as loops in $\mathcal{C}_n(M)$ ⇒ $[a] = [a'] \in B_n(M)$
- Conversely: given homotopy $a_s = \{h_i(s,\cdot)\}$:
  - assume smoothness for a moment: then, let $X_{s,t}$ = v.f. on $M \times \{t\}$ (depending smoothly on s,t) s.t. $X_{s,t}(h_i(s,t)) = \frac{\partial}{\partial s} h_i(s,t)$, & supported in a compact nbd of a_s (eg. extend to nbd around pt & cutoff fn)

    Then take φ_s = flow of this vf.
  - in general: if $|s-s'|$ small enough, then each strand of $a_{s'}$ remains in a small nbd of corresp. strand of a_s (take disjoint such nbds), then build $\varphi_{s,s'} : M \times [0,1] \circlearrowleft_{C^0}$, Id outside $\mathcal{V}(a_s)$, $\varphi_{s,s'}(a_s) = a_{s'}$; compose these ▲

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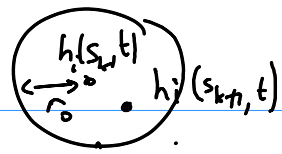
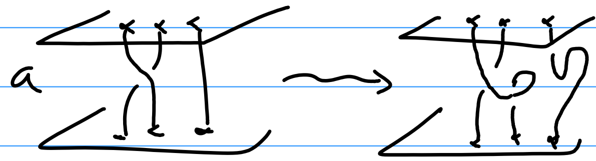
(explicitly: we're using Lebesgue number to decompose $[0,1]_s$ into finitely many substeps; in each step, build explicitly $\varphi_{s_k, s_{k+1}}$ by

outside $\bigsqcup B(h_i(s_k,t), r_0)$: Id

inside: some formula depending C^0 on target pt and $= \mathrm{Id}$ near $\partial(B(-, r_0))$



• Even better, can allow isotopies through configurations that aren't geom. braids!!



$\leadsto$ braid a'

Prop (Artin 1947) ~~for $\mathbb{R}^2$~~

2 geom. braids a, a' represent the same elt of $B_n(\Omega)$ iff $\exists$ isotopy $(\varphi_s)_{s \in [0,1]}$,

$\varphi_s \in \mathrm{Homeo}(\Omega \times [0,1])$, $\varphi_0 = \mathrm{Id}$, $\varphi_s|_{\Omega \times \{0\}} = \mathrm{Id}$, $\varphi_s|_{\Omega \times \{1\}} = \mathrm{Id}$,

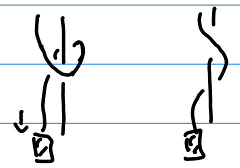
$\varphi_s(a) = a'$ (without property (*) above)

(won't prove; intuitively think: deformation of elastic strings.

Induction on # strands $\Rightarrow$ assume first k strands are always going down

Attach weight at bottom of $(k+1)^{th}$ $\leadsto$ all stuff "slides down"

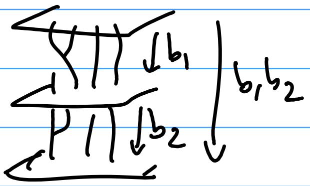
$\rightarrow$ could deform through braids.



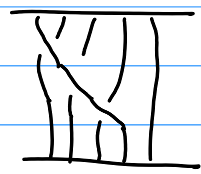
As before, thinking of a homotopy of braids or an isotopic deform^n of $\Omega \times [0,1]$ is equivalent!)

Rmk: this suggests braids are relevant to study of links!

• Product of braids = obvious:



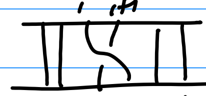
• A braid (in $\mathbb{R}^2$) can be represented by a diagram



(choose base pts $= \{1, .., n\} \in \mathbb{C}$

ensure real parts remain distinct except finitely many crossings)

$\leadsto$ in particular, get any elt of B_n as a product of $\sigma_i^{\pm 1}$, $\sigma_i =$



(counter-clockwise half twist)

$\to$ $\sigma_1, \dots, \sigma_{n-1}$ <u>generate</u> B_n (Artin)
(will see a proof later)

Also, obviously from these diagrams
$\bullet\ \sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$
$\bullet\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ (seen last time)

<u>Thm</u> (<u>Artin 1925</u>): This gives a presentation of B_n (proof later)

<u>Pure braids</u>:

• <u>Prop</u>: $\widetilde{\mathcal{C}}_n(M) \to \mathcal{C}_n(M)$ is a regular $\mathfrak{S}_n$-covering
(i.e. $\mathfrak{S}_n$ acts transitively & freely by deck transformations)

In particular the <u>pure braid group</u> $P_n(M) = \pi_1(\widetilde{\mathcal{C}}_n(M))$ is a <u>normal subgroup</u> of $B_n(M)$, and $B_n(M)/P_n(M) \simeq \mathfrak{S}_n$

$$1 \to P_n \to B_n \to \mathfrak{S}_n \to 1$$

$\to B_n \to \mathfrak{S}_n$: induced permutation of base config $\{p_1 \dots p_n\}$
P_n: pure braids = those where each strand is a closed loop in M.

Because of this structure, study the structure of $P_n(M)$ as a first step towards $B_n(M)$.
(benefit of this approach: applies to all $B_n(M)$'s)

• <u>Def</u>: Fix $Q_m = \{q_1 \dots q_m\} \subset M$ distinct pts
$\widetilde{\mathcal{C}}_n^m(M) := \widetilde{\mathcal{C}}_n(M \setminus Q_m)$ ordered configs. of n points distinct from $\{q_1 \dots q_m\}$
~~$P_n^m(M) := P_n(M \setminus Q_m) = \pi_1(\widetilde{\mathcal{C}}_n^m(M))$~~

<u>Prop</u> (Fadell-Neuwirth 1962):
$\forall r < n$, $\exists$ locally trivial fibration
$$\begin{array}{ccc} \widetilde{\mathcal{C}}_{n-r}^{m+r}(M) \xrightarrow{j} \widetilde{\mathcal{C}}_n^m(M) & & (z_1 \dots z_n) \\ \downarrow \pi & & \downarrow \\ \widetilde{\mathcal{C}}_r^m(M) & & (z_1 \dots z_r) \end{array}$$

<u>Pf</u>: • fixing $(z_1^\circ \dots z_r^\circ)$, $\pi^{-1}((z_1^\circ \dots z_r^\circ)) = \{(z_1^\circ, \dots, z_r^\circ, \underline{z_{r+1} \dots z_n}),\ z_{r+1} \dots z_n$ distinct from each other & from $q_1, \dots, q_m, z_1^\circ, \dots, z_r^\circ\}$
$\simeq \widetilde{\mathcal{C}}_{n-r}^{m+r}(M)$

• local triviality: fix neighborhoods $U_i \ni z_i^\circ$ for $1 \leq i \leq r$, disjoint from each other & from q_i's
want $\pi^{-1}(U_1 \times \dots \times U_r) \simeq U_1 \times \dots \times U_r \times \widetilde{\mathcal{C}}_{n-r}^{m+r}(M)$?
Only issue: when one (or more) of $z_{r+1} \dots z_n$, say z_k, lands in U_i

(because constraint is $z_k \neq z_i$, depends on choice of z_i)

Use a homeo of M, $\varphi_{(z_i),(z_i^0)}$ which $\begin{cases} \text{is id outside of } \sqcup U_i \\ \text{maps } U_i \to U_i, \quad z_i \mapsto z_i^0 \\ \text{depends continuously on } (z_i) \end{cases}$ → in each U_i as above

→ then $\varphi(z_k)$, $r+1 \leq k \leq n$ are distinct from each other, from q_i's, and from z_i^0's

→ get $\tilde{\mathcal{E}}_{n-r}(M \setminus \{q_1 \dots q_m, z_1^0 \dots z_r^0\})$ ✓

Thm: If $\pi_2(M - Q_m) = \pi_3(M - Q_m) = \pi_0(M - Q_m) = 1 \ \forall m \geq 0$, then

$$1 \to \pi_1(M - Q_{n-1}) \xrightarrow{j_*} P_n(M) \xrightarrow{\pi_*} P_{n-1}(M) \to 1$$

move only z_n — motions of $z_1 \dots z_n$ — motions of $z_1 \dots z_{n-1}$

Pf: This is part of the homotopy long exact seq. of fibration $\underbrace{\tilde{\mathcal{E}}_1^{n-1}(M)}_{= M - Q_{n-1}} \xrightarrow{j} \tilde{\mathcal{E}}_n(M) \to \tilde{\mathcal{E}}_{n-1}(M)$

$$\dots \to \pi_2(\tilde{\mathcal{E}}_{n-1}(M)) \to \pi_1(M - Q_{n-1}) \to \pi_1(\tilde{\mathcal{E}}_n(M)) \to \pi_1(\tilde{\mathcal{E}}_{n-1}(M)) \to \underbrace{\pi_0(M - Q_{n-1})}_{= 1 \text{ by assumption}} \to \dots$$

Lemma: if $\pi_2(M - Q_m) = \pi_3(M - Q_m) = 1 \ \forall m$ then $\pi_2(\tilde{\mathcal{E}}_n(M)) = 0 \ \forall n$

Pf: use l.e.s for $\tilde{\mathcal{E}}_{n-1}^{m+1}(M) \to \tilde{\mathcal{E}}_n^m(M) \to \tilde{\mathcal{E}}_1^m(M) = M - Q_m$ gives

$$\underbrace{\pi_3(M - Q_m)}_{= 0} \to \pi_2(\tilde{\mathcal{E}}_{n-1}^{m+1}(M)) \xrightarrow{\cong} \pi_2(\tilde{\mathcal{E}}_n^m(M)) \to \underbrace{\pi_2(M - Q_m)}_{= 0}$$

$$\Rightarrow \pi_2 \tilde{\mathcal{E}}_n(M) = \pi_2 \tilde{\mathcal{E}}_{n-1}^1(M) = \dots = \pi_2 \underbrace{\tilde{\mathcal{E}}_1^{n-1}(M)}_{M - Q_{n-1}} = 0.$$

We'll use this to get presentations of $P_n(M)$ & $B_n(M)$.

Rmk: this is only interesting if $\dim M = 2$, indeed:

Prop: M closed smooth mfld, $i: \tilde{\mathcal{E}}_n(M) \hookrightarrow M^n$

$\Rightarrow i_*: \pi_k(\tilde{\mathcal{E}}_n(M)) \to \pi_k(M)^n$ is $\twoheadrightarrow$ if $\dim M = k+1$, $\cong$ if $\dim M \geq k+2$

(Idea: complement of $\tilde{\mathcal{E}}_n(M)$ in M^n is a union of codim. d submflds $\{z_i = z_j\}$, $d = \dim M$

→ doesn't affect π_k for $k \leq d-2$

as k-spheres & homotopies $S^k \times [0,1] \to M^n$ can be made to avoid these submflds for dim' reasons)

So: $B_n(M)$ only interesting if $\dim M = 2$.

Now, see why $B_n = B_n(\mathbb{R}^2)$ plays a central role:

Consider M a surface, $\mathbb{R}^2 \simeq D \subset M$ open disc, then any configuration in $\mathbb{R}^2$ can be embedded (as a config in D) into M: incl map $\tilde{e}_n : \tilde{C}_n(\mathbb{R}^2) \hookrightarrow \tilde{C}_n(M)$

$\tilde{e}_n^m : \tilde{C}_n^m(\mathbb{R}^2) \hookrightarrow \tilde{C}_n^m(M)$ (ordered configs of n pts distinct from m given pts)

Induces $(\tilde{e}_n^m)_* : \pi_1 \tilde{C}_n^m(\mathbb{R}^2) \to \pi_1 \tilde{C}_n^m(M)$

in particular for $m=0$, $\tilde{e}_{n*} : P_n(\mathbb{R}^2) \to P_n(M)$

[Similarly after quotient by $\mathfrak{S}_n$, $e_{n*} : B_n(\mathbb{R}^2) \to B_n(M)$]

Thm (Birman). If M is a compact surface, $M \neq S^2$ or $\mathbb{RP}^2$, then $\tilde{e}_{n*}, e_{n*}$ are injective.

($\to$ can view P_n & B_n as subgroups – not normal – of $P_n(M)$, $B_n(M)$)

Pf: recall

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathbb{R}^2 - \{q_1..q_{n-1}\}) \arrow[r, "j_*"] \arrow[d, "e_*"] & P_n(\mathbb{R}^2) \arrow[r, "\pi_*"] \arrow[d, "\tilde{e}_{n*}"] & P_{n-1}(\mathbb{R}^2) \arrow[r] \arrow[d, "\tilde{e}_{n-1*}"] & 1 \\
1 \arrow[r] & \pi_1(M - \{q_1..q_{n-1}\}) \arrow[r, "j_*"] & P_n(M) \arrow[r, "\pi_*"] & P_{n-1}(M) \arrow[r] & 1
\end{tikzcd}$$

(homotopy les for fibration $\mathbb{R}^2 - \{q_1..q_{n-1}\} \hookrightarrow \tilde{C}_n \to \tilde{C}_{n-1}$ cf. above & similarly for M)

assumption $M \neq S^2$ & $\mathbb{P}^2$ gives $\pi_2(M) = \pi_2(M - \text{pts}) = 0$, needed for les to have no $\pi_2(P_{n-1}(M)) = 0$

Diagram commutes (because underlying maps commute $\tilde{e}_{n-1} \circ \pi = \pi \circ \tilde{e}_n$ embed then forget a pt or vice versa) & similarly the other one)

$M \neq S^2$ or $\mathbb{P}^2 \Rightarrow e_*$ is injective (1st vertical arrow).

Then by induction on n, $\tilde{e}_{n*}$ is injective

($n=1$ ok; $(e_{n-1})_*$ inj $\Rightarrow \ker(\tilde{e}_{n*}) \subset \pi_*^{-1}(\ker \tilde{e}_{n-1*}) = \ker \pi_* = \operatorname{Im} j_*$

but then $j_* e_* = \tilde{e}_{n*} j_*$ must be injective ✓

Hence $P_n(\mathbb{R}^2) \hookrightarrow P_n(M)$.

- similarly,

$$\begin{tikzcd}
1 \arrow[r] & P_n(\mathbb{R}^2) \arrow[r] \arrow[d, "\tilde{e}_{n*}"] & B_n(\mathbb{R}^2) \arrow[r] \arrow[d, "e_{n*}"] & \mathfrak{S}_n \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & P_n(M) \arrow[r] & B_n(M) \arrow[r] & \mathfrak{S}_n \arrow[r] & 1
\end{tikzcd}$$

commutes $\rightsquigarrow e_{n*}$ injective too. ∎

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2 phenomena for braids in M: – "classical" braiding inside a disc – pts can move around M. $\Big] =$ in a way only possibilities

Thm (Goldberg 1973): M closed surface $\neq S^2$ or $\mathbb{P}^2$, $\tilde{e}_{n*} : P_n(\mathbb{R}^2) \to P_n(M)$ (seen: inj.)

$i_* : P_n(M) = \pi_1 \tilde{C}_n(M) \to (\pi_1 M)^n$ (seen: surj.)

[won't prove] $\ker i_* =$ normal closure of $\operatorname{im}(\tilde{e}_{n*})$