Lecture 18 – Mon Apr 24

Generalize braid monodromy to

- <u>projective curves</u>: $C \subset \mathbb{CP}^2$ of degree d (recall $\mathbb{CP}^2 = \mathbb{C}^3 - 0/\mathbb{C}^* = \mathbb{C}^2 \cup (\mathbb{CP}^1_\infty)$)

  (C: $P(x,y,z) = 0$ homogeneous poly of deg. d)

- Consider the projection $\pi: \mathbb{CP}^2 - \{(0:0:1)\} \to \mathbb{CP}^1$

  $$(x:y:z) \longmapsto (x:y)$$

  Fibers of π are $\simeq \mathbb{C}$ (lines through $(0:0:1)$, minus that point)

  π realizes $\mathbb{CP}^2$-pt $\cong$ complex line bundle of degree 1 ("$\mathcal{O}(1)$") over $\mathbb{CP}^1$
  = dual of tautological line bundle

  (• An element $(x:y:z) \in \pi^{-1}(x:y)$ $\leftrightarrow$ a linear form on the tautological line corresp to $(x:y)$ inside $\mathbb{C}^2$: indeed, to an elt (x',y') of that line, associate $z' \in \mathbb{C}$ st. $(x':y':z') = (x:y:z)$.

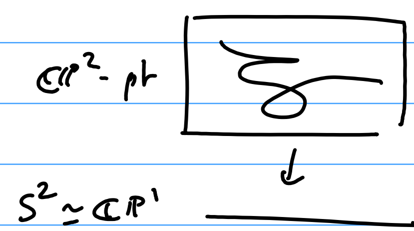
  • zero section = the line $\{z=0\} = \{(x:y:0)\} \simeq \mathbb{CP}^1 \subset \mathbb{CP}^2$

  • lines $\not\ni (0:0:1)$ are sections of π; these sections vanish once, at pt of intersection with the line $\{z=0\}$ – indeed degree is 1).

- Assume $(0:0:1) \notin C$

  Then $\pi_{|C}: C \to \mathbb{CP}^1$ has degree d

  (<u>note</u>: total intersection number b/w C and any proj line is d)



  Consider again $\{q_1, \dots, q_r\} \in \mathbb{CP}^1$ = set of pts s.t. $|\pi^{-1}(q_i) \cap C| < d$.
  (projections of sing. pts or vert. tangencies)

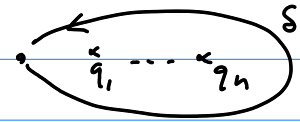
  To each $(x:y) \in \mathbb{CP}^1 - \{q_1, \dots q_r\}$, associate $\pi_{|C}^{-1}(x:y) = d$ points in $\mathbb{C}$

  $\to$ get braid monodromy out of this?

  ⚠ To define braid monodromy, really need all fibers of π to be "<u>the same</u> $\mathbb{C}$"
  ie. requires choice of a trivialization of the line bundle $\pi: \mathbb{CP}^2\text{-pt} \to \mathbb{CP}^1$.
  Problem: such a triv$^{\text{n}}$ doesn't exist over entire $\mathbb{CP}^1$.
  So: discard a point of $\mathbb{CP}^1 - \{q_1, \dots q_r\}$ (call it "∞") and work over $\mathbb{CP}^1 - \{\infty\} \simeq \mathbb{C}$. [by connectedness, it doesn't matter which pt we discard]

  There we have braid monodromy $\rho: \pi_1(\mathbb{C} - \{q_1, \dots q_r\}) \longrightarrow B_d$

  In fact this is equivt to restricting to $\pi^{-1}(\mathbb{C}) \simeq \mathbb{C}^2$ and braid m'y of $(C \cap \mathbb{C}^2)$

- As before, ρ is def$^{\underline{d}}$ up to conj. by an element of B_d.
  (choice of identification b/w reference config. $\pi_{|C}^{-1}(*) \subset C$ and $\{1,..,n\} \subset \mathbb{C}$).
- monodromy around each special pt gives information about local behavior of C there.
- monodromy at ∞:  $\longmapsto \rho(\delta) = ?$

<u>Lemma:</u> $\rho(\delta) = \Delta^2 \in B_d$.

<u>Pf:</u> Observe: δ = meridian around $\infty \in \mathbb{CP}^1$
and in fact nothing happens to C near $\pi^{-1}(\infty)$
$\rightsquigarrow$ if it weren't for obstruction to trivialization, would have $\rho(\delta)=1$.

Trivialization of π over $\mathbb{C} = \{(x:y), y\neq 0\} \subset \mathbb{CP}^1$ is given by
$(x:y:z) \leftrightarrow \left(\frac{x}{y}, \frac{z}{y}\right)$ $\quad {}^{\shortparallel}(\frac{x}{y}:1)$
$\shortparallel \quad y\neq 0$
$(\frac{x}{y}:1:\frac{z}{y})$

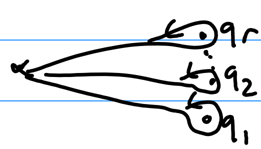
The points of $C \cap \pi^{-1}(1:0)$ are of the form $(1:0:z_i)$, $z_1..z_d \in \mathbb{C}$ distinct

Transversality $\rightarrow$ $\pi^{-1}(Re^{i\theta}:1) \cap C$
$(R>>1)$ ${}^{\shortparallel}(1:\frac{1}{R}e^{-i\theta})$ consists of pts $(1:\frac{1}{R}e^{-i\theta}:\tilde{z}_i(\theta))$
$\uparrow$ close to z_i
which under trivialization $\sim Re^{i\theta}\tilde{z}_i(\theta)$

Hence braid monodromy is full rotation by 2π, i.e. Δ^2. ∎

<u>Braid monodromy factorization:</u> $\rho: \underbrace{\pi_1(\mathbb{C}-\{q_1,..q_r\})}_{\text{free group}} \to B_d$

$\rightarrow$ choose r generators $\gamma_1, .., \gamma_r$ such that $\gamma_1 \cdots \gamma_r = \delta$

 and let $b_i = \rho(\gamma_i)$

Then above lemma says: $\Delta^2 = \prod_{i=1}^{r} b_i$.

$\rightarrow$ express the braid monodromy of C by a <u>factorization of Δ^2</u> into the individual monodromies at $q_1..q_r$. (i.e. a tuple $(b_1..b_r)$ s.t. $\prod b_i = \Delta^2$)

Of course this depends on choice of generating set $\gamma_1, \dots \gamma_r$.

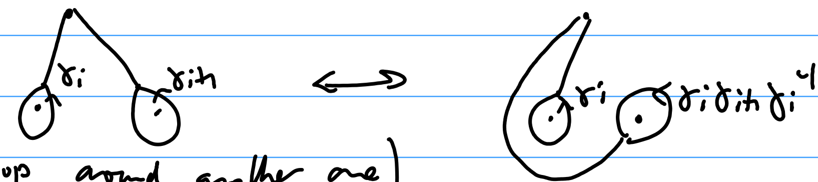
(<u>Rmk</u>: just want: each γ_i goes around one q_i – labelling of q_i isn't fixed in advance – and $\prod \gamma_i = \delta$)

Def: Hurwitz equivalence = equiv relation on tuples of elts in a group G (here $G = B_d$) generated by "Hurwitz moves"

$$(b_1, \dots, b_r) \longleftrightarrow (b_1 \dots, b_i b_{i+1} b_i^{-1}, b_i, \dots, b_r)$$

(& inverse move $(b_1, \dots, b_{i+1}, b_{i+1}^{-1} b_i b_{i+1}, \dots, b_r)$)

This corresponds to



(slide a loop around another one).

Lemma: any 2 generating sets for $\pi_1(\mathbb{C} \setminus \{q_1 \dots q_r\})$ are related by Hurwitz moves

Pf:
- any 2 generating sets $(\gamma_1, \dots, \gamma_r)$ & $(\gamma'_1 \dots \gamma'_r)$ are related by action of a braid $\in B_r \circlearrowright \pi_1(\mathbb{C} \setminus \{q_1 \dots q_r\})$

  Indeed, we can define an autom. of the free gp $\pi_1(\mathbb{C} \setminus \{q_i\}) \to \pi_1$, $\gamma_i \mapsto \gamma'_i$

  It maps generators γ_i to conjugates of generators (each γ'_i is conj. to one of $\gamma_1, \dots, \gamma_r$), and $\prod \gamma_i$ to itself.

  $\Rightarrow$ by a thm of Artin (seen a while ago), this autom. is in the image of $B_r \hookrightarrow \mathrm{Aut}(F_r)$.

- now, B_r is generated by $\sigma_1, \dots \sigma_{r-1}$, and

  $(\sigma_i)_*$: $\gamma_i \mapsto \gamma_i \gamma_{i+1} \gamma_i^{-1}$
  $\gamma_{i+1} \mapsto \gamma_i$
  (others unchanged)

  corresponds exactly to a Hurwitz move.

  $\Rightarrow (\gamma_1 \dots \gamma_r)$, $(\gamma'_1 \dots \gamma'_r)$ always related by seq. of H-moves $\blacktriangle$

Rmk: so, in fact, Hurwitz equivalence corresponds to orbits of a braid group action: B_r acts on $G^r = \mathrm{Hom}(F_r, G)$

$\uparrow$ free gp on r letters. B_r acts

(Here $G = B_d$, but what we discussed doesn't rely on it)

Finally, to a proj. curve of degree d, we associate $(b_1, \dots, b_r) \in B_d$ s.t. $\prod b_i = \Delta^2$, up to simult. conjugation by an elt of B_d & Hurwitz moves (i.e. $B_r \times B_d$-action on $\mathrm{Hom}(F_r, B_d)$).

• Further generalization: Hurwitz curves

The constr$^{\text{n}}$ of braid monodromy doesn't require C to be an algebraic curve...

Def: $C \subset \mathbb{CP}^2$ closed oriented $\dim_{\mathbb{R}} 2$ submfld w/ isolated singularities is a <u>Hurwitz curve</u> if

- $(0:0:1) \notin C$
- C intersects transversely & positively the fibers of $\pi: (x:y:z) \mapsto (x:y)$ except at finitely many pts $p_1, \dots, p_r \in C$ (singularities & vertical tangencies)

• Given any Hurwitz curve $C \subset \mathbb{CP}^2$, we can still define braid monodromy as above. The "degree" of C is $d = [C] \cdot [\text{line}] > 0$ (intersection number b/w 2 classes in $H_2(\mathbb{CP}^2, \mathbb{Z}) \simeq \mathbb{Z}$) (ie. $[C] = d \cdot [\text{line}]$)

$\rightsquigarrow$ factorization in B_d.

• Usually one requires a bit more, by prescribing a class of model behaviors at p_i:

- near each p_i, $\exists$ nbd U_i, a model curve $\tilde{C}_i \subset \mathbb{C}^2$ (in allowed class of models) and orientation-preserving local diffeos. s.t.

$$\begin{array}{ccc}
(U_i \cap C) & \xrightarrow{\sim} & \tilde{C}_i \\
\cap & & \cap \\
U_i & \xrightarrow{\sim} & \mathbb{C}^2 \\
\pi \downarrow & & \downarrow pr_1 \\
\pi(U_i) & \xrightarrow{\sim} & \mathbb{C}
\end{array}$$