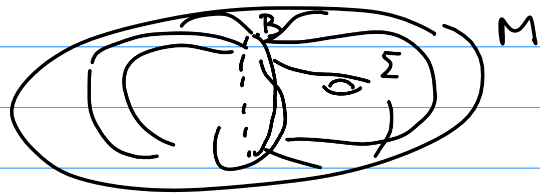
Lecture 15 — April 10

3-manifolds & open books

Def: An open book decomposition of M^3 (closed oriented) is a pair (B, π),
- $B \subset M^3$ oriented link — BINDING
- $\pi : M - B \to S^1$ fibration s.t. $\forall \theta$, $\pi^{-1}(\theta) = \mathring{\Sigma}_\theta$ for some surface with $\partial \Sigma_\theta = B$ — PAGE



M = "spinning" Σ around the binding B.
Return map for a v.f. transverse to pages gives "monodromy" $\in \mathrm{Diff}^+(\Sigma)$; can choose Id near B.

Abstractly, an open book is characterized by
- Σ = surface w/ boundary
- $\phi \in \mathrm{Diff}^+(\Sigma, \partial\Sigma)$ MONODROMY

(up to isotopy, so $[\phi] \in \mathrm{Map}(\Sigma)$
· up to conjugation by a diffeo h of Σ) actually

Construction: given (Σ, ϕ), • mapping torus of ϕ: $\Sigma \times [0,1]/\sim$ $\quad (x,1) \sim (\phi(x), 0)$ $\forall x \in \Sigma$

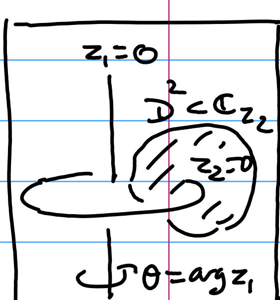
this is a 3-mfld with boundary $\partial\Sigma \times S^1$ (union of tori)
It fibers over S^1 with fiber Σ ($\Leftrightarrow$ all the pages, unbound)

- $M(\Sigma, \phi) = \left(\Sigma \times [0,1]/\sim\right) \cup_\partial \left(\partial\Sigma \times D^2\right)$ closed 3-mfld

($\Leftrightarrow$ glue pages together along the binding).

Lemma: (M, B, π) up to diffeomorphism $\overset{1-1}{\Longleftrightarrow}$ (Σ, ϕ) up to isotopy & diffeo.

Example: $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$



$U = \{z_1 = 0\} \cap S^3 = \{0\} \times S^1$ unknot
$p : S^3 - U \to S^1$
$(z_1, z_2) \mapsto z_1/|z_1|$
Page $= D^2$ (e.g. for $\theta = 0$, corresponds to $\begin{cases} z_2 \in D^2 \\ z_1 \in \mathbb{R}_+,\ z_1 = \sqrt{1-|z_2|^2} \end{cases}$
Monodromy = Id.

Corresponds to: S^3 = union of 2 solid tori, w/ cores the 2 linked unknots $S^1 \times \{0\}$ (core of the mapping torus) & $\{0\} \times S^1 = U$ (the binding)

Thm (Alexander 1920): ‖ Every closed oriented 3-mfld admits an open book decomposition

Many proofs; one uses the following 2 ingredients:

① **Thm (Alexander 1920):**

‖ Every closed oriented 3-manifold is a branched cover of S^3 branched along some link $L \subset S^3$.

(in fact, has been refined to show: M can be realized as a 3-fold branched cover).

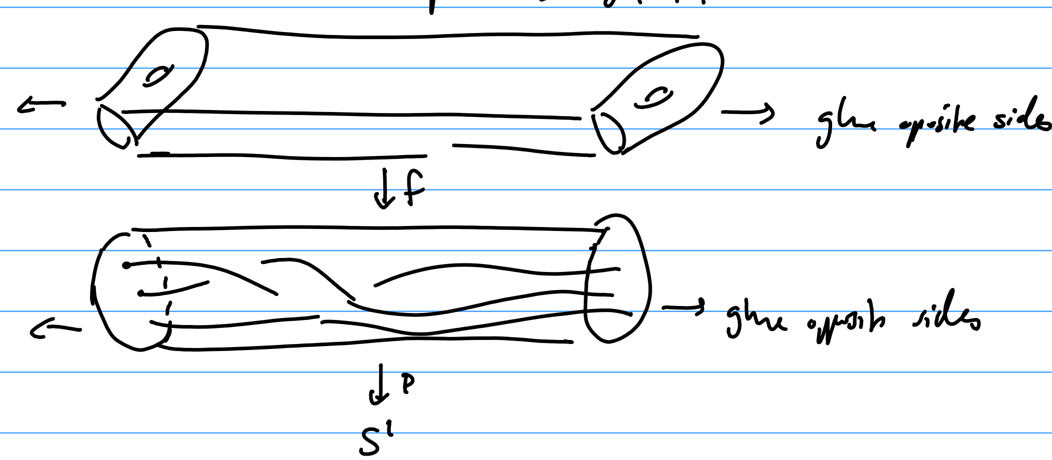
② (Alexander) any link $L \subset S^3$ can be made into a closed braid.

Then: M = branched cover of S^3 along L = closed braid with axis: the unknot $U \subset S^3$ above.

This means L is everywhere transverse to the pages $D^2_\theta \subset S^3$.

Consider $M \xrightarrow{f} S^3$ and lift the above open book on S^3, i.e. let $B = f^{-1}(U)$, it's a link in M (since f is unramified above U).
(U = unknot)

$\pi = p \circ f : M - B \to S^1$ is a fibration (using that branch set of $f \pitchfork$ fibers of p) & defines an open book on M



- The page Σ of the open book π is a branched cover of D^2 (branched at # pts = #strands of the braid β chosen to represent L)
- The homeomorphism of (D^2, pts) induced by β is liftable and its lift is precisely the monodromy of the open book on M. ⊠

This is very explicit and algorithmic!

- In light of Markov's thm: can modify a closed braid by conjugations & stabilizations. What does this do to the open book?

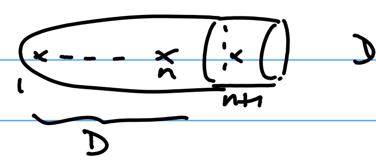
- <u>conjugation</u>: does nothing. The mapping torus of φ & the mapping torus of $\tilde{\varphi} = h\varphi h^{-1}$ are diffeomorphic (by $\Sigma\times[0,1]/\sim \longrightarrow \Sigma\times[0,1]/\sim$, $(x,t) \longmapsto (h(x),t)$)

  & $\varphi, \tilde{\varphi}$ define equivalent open books.

- <u>stabilization</u> is more interesting!

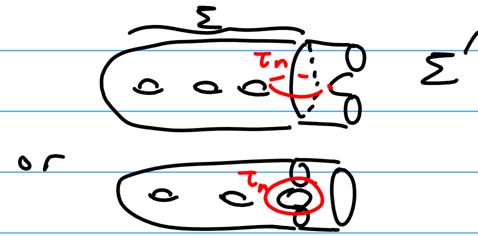
  start with $\beta \in B_n$ and a branched cover $\Sigma \longrightarrow D^2$ branched at n points. Assume for simplicity it's a double cover.

  Let D' = larger disc w/ $n+1$ points in it

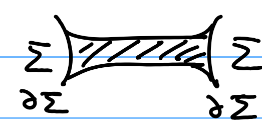


Consider Σ' = double cover of D',

$\Sigma' = \Sigma \cup$ 1-handle.



In both cases, topologically Σ' = attach

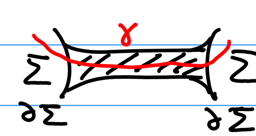


Now, $\beta \in B_n \subset B_{n+1}$ (incl. $D\subset D'$) $\overset{\text{lifting}}{\longmapsto} \phi \in Map(\Sigma) \subset Map(\Sigma')$ (inclusion $\Sigma\subset\Sigma'$)

$\sigma_n \overset{\text{lifting}}{\longmapsto}$ Dehn twist τ_n as pictured above.

Hence $\beta' = \beta.\sigma_n^{\pm 1} \longmapsto \phi.\tau_n^{\pm 1}$

More intrinsically: τ_n = Dehn twist along a curve γ passing through new handle



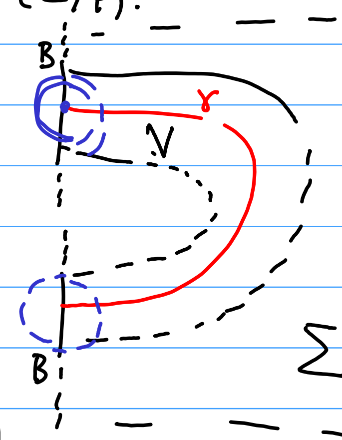
(<u>Note</u>: things are up to conjugation by $Map(\Sigma)$ → it doesn't really matter how γ is chosen exactly)

Even if the covering $\Sigma \to D^2$ has more than 2 sheets, one can still think of stabilization in terms of attaching handles to Σ & adding Dehn twists to the monodromy.

Def: Given an open book (Σ, ϕ), a <u>positive</u> (resp. neg) <u>stabilization</u> of (Σ, ϕ) is an open book (Σ', ϕ')
- $\Sigma' = \Sigma \cup$ 1-handle
- $\phi' = \phi \cdot \tau_\gamma^{\pm 1}$, γ = s.c.c. on Σ' passing exactly once through the handle.

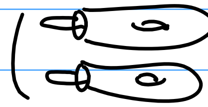
Prop: (Σ', ϕ') stab^n of (Σ, ϕ)
$\Rightarrow M(\Sigma, \phi)$ and $M(\Sigma', \phi')$ are diffeomorphic.

Idea: $M' = M(\Sigma', \phi')$ is obtained by a "surgery" operation on $M = M(\Sigma, \phi)$:



① attach handles to the pages
This takes place <u>near 2 points of binding B</u>
(& doesn't modify the rest of M)

② modify the monodromy by adding τ_γ
= takes place near a given page Σ



(↕ replace $\Sigma \times (-\varepsilon, \varepsilon)$ by something that twists along γ)

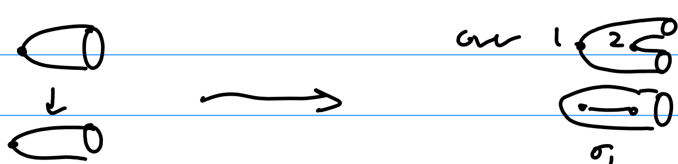
and more precisely, along a copy of arc γ inside it!

So: M' is obtained from M by deleting a small nbd of an embedded arc (joining 2 pts of B) & pasting something into there.
(and what is pasted is "universal" - doesn't depend on M).

This is a surgery operation on a small ball $\Rightarrow$ it's a <u>connected sum</u> with some 3-manifold N (N = what we obtain if we perform the surgery starting with S^3)

But in case of double covers / Markov stabilization we know that nothing changes: e.g. $S^3 = M(D^2, Id) \rightsquigarrow M(\text{[annulus]}, \tau^{\pm 1})$

2:1 cover



= 2:1 cover of S^3 branched at closure of $\sigma_1^{\pm 1}$ i.e. at unknot

, still S^3.

(S^3 = double cover of S^3 br. at unknot: $(z_1, z_2) \mapsto (z_1^2, z_2)$ + rescale first coord. to land in sphere)

So the operation is trivial ◆

Open books are related to <u>Contact structures</u> (↔ odd-dim'l counterparts of symplectic structures if you know these)

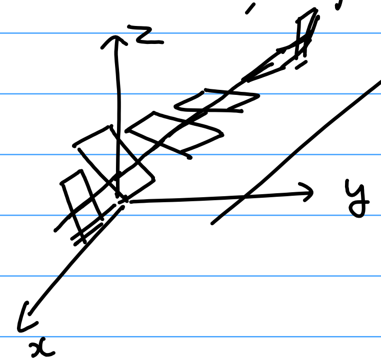
Def: A <u>contact structure</u> on an oriented mfld M^{2n+1} is an oriented hyperplane field ξ that is maximally non-integrable, i.e. $\exists$ 1-form α (<u>contact form</u>) s.t. $\xi = \ker\alpha$ (w/ orientation) and $\alpha\wedge(d\alpha)^n > 0$.

(in dim 3: $\alpha\wedge d\alpha > 0$).

($d\alpha_x$ is a symplectic form on each contact plane ξ_x, $x\in M$)

<u>E.g.:</u> $\mathbb{R}^3$, $\alpha_0 = dz + x\,dy$

$\xi_0 = \ker\alpha_0$



• <u>Darboux thm:</u> every (M^3,ξ) is locally diffeo to $(\mathbb{R}^3,\xi_0)$

(i.e. $\exists$ loc coords where $\xi = \ker(dz + x\,dy)$)

• isotopic contact structures are equivalent: <u>Gray's theorem</u>:

<u>Thm:</u> (M^3,ξ_t) $t\in[0,1]$ contact str on a <u>closed</u> mfld $\Rightarrow \exists\ \phi_t$ diffeos s.t. $\phi_t^*\xi_t = \xi_0$.

• The various contact forms defining a given ξ are all proportional:

$\xi = \ker\alpha = \ker\alpha' \iff \alpha' = f\alpha$ for some $f: M\to\mathbb{R}_+$

$(\alpha'\wedge d\alpha' = f^2\alpha\wedge d\alpha)$

• <u>Ex:</u> S^3, $\xi_{std} = TS^3\cap J(TS^3)$ max. complex subspace of $TS^3\subset T(\mathbb{C}^2)$

$\xi_{std} = \ker\left(\frac{1}{2}(x_1\,dy_1 - y_1\,dx_1 + x_2\,dy_2 - y_2\,dx_2)\right)$ α_{std}, $d\alpha_{std} = \omega_0$

More generally, a "J-convex" hypersurface in a complex mfld inherits a contact structure by looking at max. complex directions in its tangent space.

Even more generally, "contact-type hypersurfaces" in symplectic mflds (see later).

M contact, $\xi = \ker d\alpha$ $\iff$ $M\times\mathbb{R}$, $\omega = d(e^t\alpha) = e^t(dt\wedge\alpha + d\alpha)$ is symplectic (i.e. ω closed nondegenerate 2-form)

Thm (Giroux, 2000):

$\{\text{contact structs on } M^3\}/\text{isotopy} \xleftrightarrow{1-1} \{\text{open book decomps. of } M^3\}/\text{positive stabilization}$