Lecture 14 – April 5

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The lifting homomorphism:

$f: \Sigma \xrightarrow[n:1]{} \Sigma'$ covering branched at $\{q_1, \dots, q_m\} \subset \mathring{\Sigma}'$; assume Σ, Σ' <u>not</u> closed.

Choose a base pt $b \in \partial\Sigma'$, & consider the monodromy homom. $\theta: \pi_1(\Sigma' \setminus \{q_i\}) \to \mathfrak{S}_n$
& labelling of sheets (i.e. $f^{-1}(b) = \{b_1, \dots, b_n\}$)

$\mathrm{Map}_{\{m\}}(\Sigma')$ acts on $\pi_1(\Sigma' \setminus \{q_i\})$; hence also on $\mathrm{Hom}(\pi_1(\Sigma' \setminus \{q_i\}), \mathfrak{S}_n)$ by composition on the right.

↳ homeos fixing $\partial\Sigma'$ and mapping set $\{q_i\}$ to itself.

Def: $\phi \in \mathrm{Homeo}^+_{\{n\}}(\Sigma')$ is liftable if $\exists\, \tilde\phi \in \mathrm{Homeo}^+(\Sigma)$ s.t.
$$\begin{array}{ccc} \Sigma & \xrightarrow{\tilde\phi} & \Sigma \\ f\downarrow & & \downarrow f \\ \Sigma' & \xrightarrow{\phi} & \Sigma' \end{array}$$
commutes

Rmk: if $\tilde\Phi$ exists then it is unique (assuming Σ' connected)

Prop: ϕ is liftable iff induced action on $\pi_1(\Sigma' \setminus \{q_i\})$ satisfies $\theta \circ \phi_* = \theta$.

- We have a <u>lifting homomorphism</u> associated to the branched cover:

$$\mathrm{Map}_{\{m\}}(\Sigma') \supset \text{Liftable} \longrightarrow \mathrm{Map}(\Sigma)$$

- In the case of double covers of D^2 and S^2 considered above, every mapping class is liftable! Indeed, monodromy @ each branch pt is $(1\,2)$, so θ maps γ to $(1\,2)$ if total rotation # of γ around q_i's is odd, Id if even, & this property is clearly preserved under composition with ϕ_* $\forall \phi \in \mathrm{Homeo}^+$

• Hence, get important lifting maps

$$\begin{cases} B_{2g+1} = \mathrm{Map}_{\{2g+1\}}(D^2) \longrightarrow \mathrm{Map}(\Sigma_{g,1}) \\ B_{2g+2} = \mathrm{Map}_{\{2g+2\}}(D^2) \longrightarrow \mathrm{Map}(\Sigma_{g,2}) \end{cases}$$

Rmk: This is neither injective nor surjective in general; the image = those homeos of Σ which commute with the involution of the double cover. (easy to see from the construction)

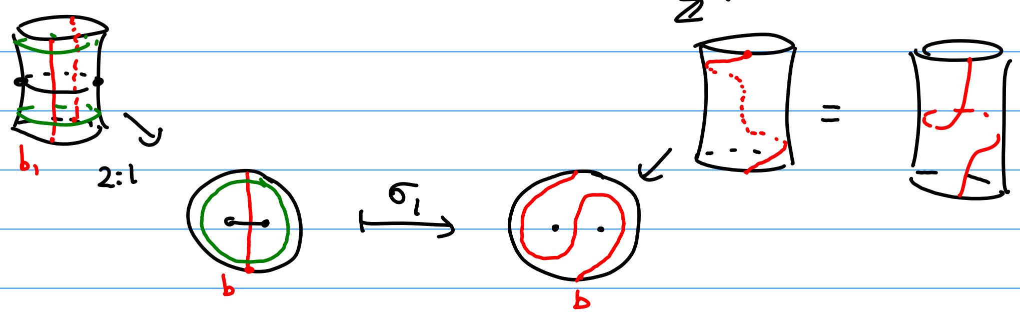
This is known as the hyperelliptic subgroup of $\mathrm{Map}(\Sigma)$.

(in Riem. surf. theory, a closed complex curve is hypell. if it has a line bundle of deg. 2 w/ 2 indep't sections; this is equiv to being holomorphically a double cover of S^2; hyp. MCG = commutator of hyp. involution)

This construction will explain a lot of the features of Map_g.

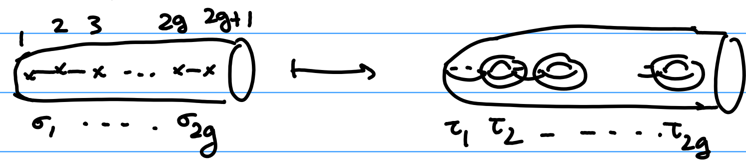
• If Σ, Σ' are closed surfaces, then there's also a notion of lifting, but not as nice — need to choose a representative $\phi \in \mathrm{Homeo}^+_{\{m\}}(\Sigma')$ which fixes some given base pt (distinct from the q_i's), and depending on choice of base point in Σ' or of representative ϕ for a given base pt, may get lifts $\tilde{\Phi}$ which differ by deck transformations of the covering. (so there isn't a canonical lift). E.g. for $\Sigma_g \xrightarrow{2:1} S^2$, no nat'l homom. to Map_g from $\mathrm{Map}_{\{2g+2\}}(S^2)$ ($=$ quotient of B_{2g+2}), rather from an extension by $\mathbb{Z}/2$; though $B_{2g+2} \to \mathrm{Map}_{g,2} \downarrow \mathrm{Map}_g$

• Lifting half twists: consider the lifting map $B_2 \to \mathrm{Map}(\Sigma_{0,2}) \cong \mathbb{Z}$, $B_2 \cong \mathbb{Z}$, $\mathrm{Map}_{0,2} \to \mathrm{Map}_g$



Hence half twist along an arc b/w 2 branch pts $\longmapsto$ Dehn twist / the closed curve which lies above it.

In particular,



(similarly for the $\Sigma_{g,2}$ case)

Next, go back to understanding mapping class groups ...

Some relations among Dehn twists:

- **braid relations:** $|\alpha\cap\beta| = 0 \Rightarrow \tau_\alpha\tau_\beta = \tau_\beta\tau_\alpha$ ← obvious

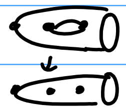
  $|\alpha\cap\beta| = 1 \Rightarrow \tau_\alpha\tau_\beta\tau_\alpha = \tau_\beta\tau_\alpha\tau_\beta$ ← equiv to:

  $$\underbrace{\tau_\alpha\tau_\beta\tau_\alpha^{-1}}_{\tau_{\tau_\alpha(\beta)}} = \underbrace{\tau_\beta^{-1}\tau_\alpha\tau_\beta}_{\tau_{\tau_\beta^{-1}(\alpha)}}$$

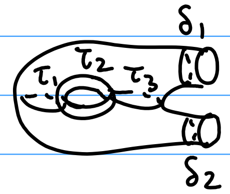
  follows from $\tau_\alpha(\beta) = \tau_\beta^{-1}(\alpha)$

  NB: $V(\alpha\cup\beta) = \Sigma_{1,1}$ so this is really a relation inherited from $\mathrm{Map}_{1,1}$ under inclusion maps for arbitrary handles $\Sigma_{1,1} \subset \Sigma_{g,r}$.

  - Can also be thought of as lifting of relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ in B_3 under double cover



- **chain relation:** in $\mathrm{Map}_{1,2}$, $(\tau_1\tau_2\tau_3)^4 = \delta_1\delta_2$

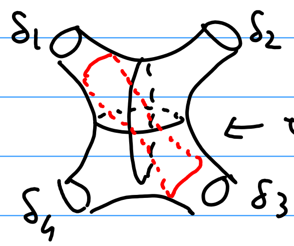


Can check using lifting $B_4 \longrightarrow \mathrm{Map}_{1,2}$

In B_4, $(\sigma_1\sigma_2\sigma_3)^4 = \Delta^2 =$ Dehn twist / boundary of D^2

- covering is trivial near boundary → boundary twist lifts to boundary twist in each sheet of the covering, ie. $\Delta^2 \mapsto \delta_1\delta_2$ ✓

- **lantern relation:** in $\mathrm{Map}_{0,4}$, $\tau_1\tau_2\tau_3 = \delta_1\delta_2\delta_3\delta_4$

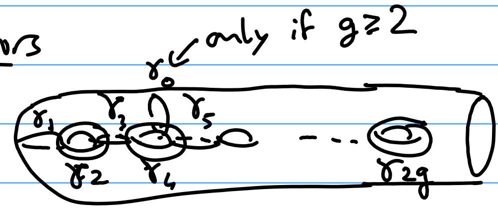


← τ_i = 3 different ways of separating 2+2 punctures

Can't explain easily in terms of lifting ... we'll see later a geometric argument. (Meanwhile, could check these by comparing actions of LHS & RHS on loops & arcs generating π_1 and π_1 rel. boundary)

Thm (Wajnryb 1983): $\forall g \geq 1$, $\mathrm{Map}_{g,1}$ admits a presentation with generators

$\tau_0, \tau_1, \dots, \tau_{2g}$
Dehn twists along $\gamma_0 \dots \gamma_{2g}$



and relations. (i) braids: $\tau_i\tau_j=\tau_j\tau_i$ if $\gamma_i\cap\gamma_j=\phi$
$\tau_i\tau_j\tau_i=\tau_j\tau_i\tau_j$ if $|\gamma_i\cap\gamma_j|=1$.

(ii) (if $g\geq 2$): one chain:

$$(\tau_1\tau_2\tau_3)^4=\tau_0\cdot\underbrace{(\tau_4\tau_3\tau_2\tau_1\tau_1\tau_2\tau_3\tau_4)^{-1}\tau_0(\tau_4\tau_3\tau_2\tau_1\tau_1\tau_2\tau_3\tau_4)}_{\bar\tau_0}$$

(iii) (if $g\geq 3$): one lantern: $\tau_0\xi_1\xi_2=\tau_1\tau_3\tau_5\xi_3$

where $\xi_1=(\tau_4\tau_3\tau_5\tau_4)\,\tau_0\,(\tau_4\tau_3\tau_5\tau_4)^{-1}$

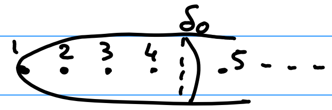
$\xi_2=(\tau_2\tau_1\tau_3\tau_2)\,\xi_1\,(\tau_2\tau_1\tau_3\tau_2)^{-1}$

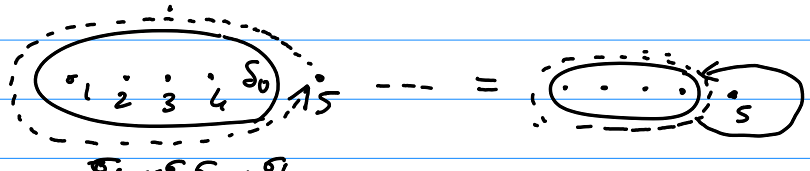
$\xi_3=(\tau_4^{-1}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}u\,\tau_2\tau_3\tau_4\tau_5\tau_6)^{-1}\,\tau_0\,(\tau_4^{-1}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}u\,\tau_2\tau_3\tau_4\tau_5\tau_6)$

where $u=(\tau_5\tau_6)\,\xi_1\,(\tau_5\tau_6)^{-1}$

What are the various Dehn twists in there?

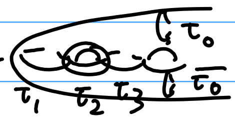
• $\bar\tau_0$ = Dehn twist about $\underbrace{(\tau_4\cdots\tau_1\tau_1\cdots\tau_4)}_{\text{lift of } \sigma_4\cdots\sigma_1\sigma_1\cdots\sigma_4}(\gamma_0)$ → one of the 2 lifts of δ_0





$\sigma_4\cdots\sigma_1\sigma_1\cdots\sigma_4$

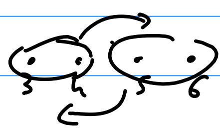
→ get the other lift of δ_0 i.e.:



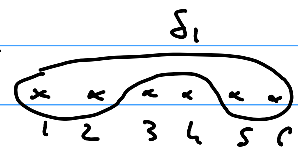
→ (ii) is indeed a chain relation (with $\bar\tau_0$ expressed in terms of the generators)

• using the same idea:

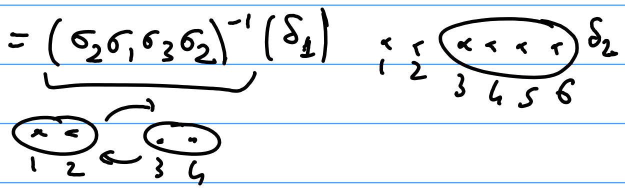
ξ_1 is a twist about a lift of $\delta_1=\underbrace{(\sigma_4\sigma_3\sigma_5\sigma_4)^{-1}}(\delta_0)$

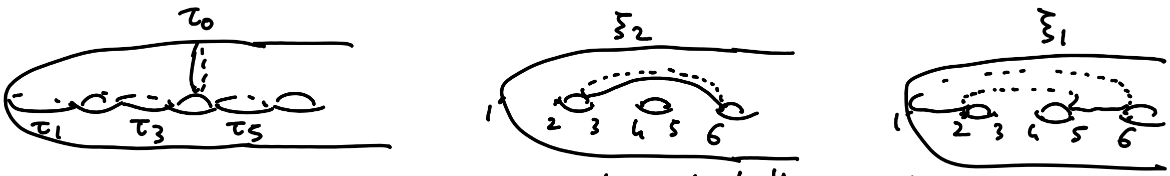


get →

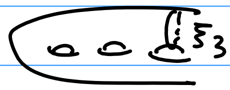


ξ_2 is a twist about a lift of $\delta_2=\underbrace{(\sigma_2\sigma_1\sigma_3\sigma_2)^{-1}}(\delta_1)$



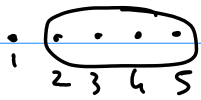
So 

(can check these are the correct lifts)

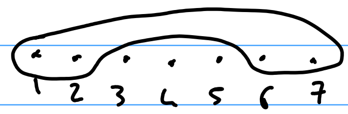
We have most of the lantern ... still need to show $\xi_3 =$ 

Proceed in stages:

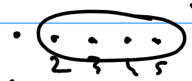
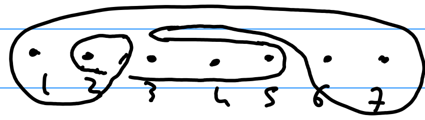
- Conjugate τ_0 by $\tau_4^{-1}\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}$ → get Dehn twist about a lift of

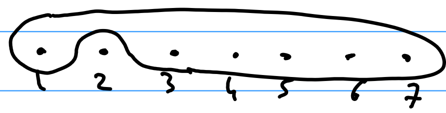
  $(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1})(\delta_0) =$ 

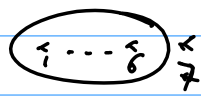
  (⚠ $(\sigma\sigma')(\delta) = \sigma'(\sigma(\delta))$)

- u = twist about a lift of $(\sigma_5\sigma_6)^{-1}(\delta_1) =$ 

- These 2 s.c.c's intersect <u>once</u> → $u^{-1}(\dots)u$ = Dehn twist about

  $u($  $) =$ 

  $=$ 

- then conj. by $\tau_2\tau_3\tau_4\tau_5\tau_6$ ⇒ get Dehn twist / a lift of 

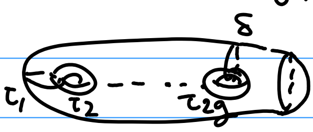
  = as claimed.

So the claimed relations do hold; we want try to show they are a complete set of defining relations for $\mathrm{Map}_{g,1}$ ---

• Thm (Wajnryb):

Map_g admits a presentation w/ same generators $\tau_0, \dots, \tau_{2g}$, and relations: (i), (ii), (iii) as above, and

(iv) $[\tau_{2g}\tau_{2g-1}\cdots\tau_2\tau_1\tau_1\tau_2\cdots\tau_{2g}, \delta] = 1$,

where $\delta =$  expressed as a word in $\tau_0, \tau_1, \dots, \tau_{2g}$.

NB: same arg^t as for chain relation ⇒ $(\dots)\delta(\dots)^{-1} = \bar{\delta}$ twist about other lifts, then $\delta = \bar{\delta}$ obvious

(the expression is rather evil - do what we did for ξ_3 inductively to get $\tau_0 \to \xi_3 \to \dots \to \delta$.