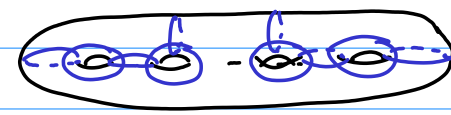
Mon Apr 3 – Lecture 13

<u>Last time</u>: we saw $\mathrm{Map}_g = \mathrm{Map}(\Sigma_g)$, $\mathrm{Map}_{g,r} = \mathrm{Map}(\Sigma_{g,r})$ are gen^d by Dehn twists.

Actually, they admit <u>finite presentations</u> w/ Dehn twists as generators.

- <u>Lickorish, 60's</u>: Map_g is finitely generated – using $3g-1$ Dehn twists



<u>Main idea</u>: since Map_g is gen^d by Dehn twists, enough to show that the Dehn twist about any simple closed curve γ can be expressed in terms of these $3g-1$ generators. Prove this by induction on how "complicated" γ is, i.e. the sum of its geometric intersection numbers with these curves + some others (separating Σ_g into pairs of pants).

<u>Key point</u>: $\| \forall h \in \mathrm{Homeo}^+, \forall \gamma$ s.c.c., $h \circ T_\gamma \circ h^{-1} = T_{h(\gamma)}$.

→ it's enough to find h expressible in terms of Dehn twists about curves simpler than γ, s.t. $h(\gamma)$ is simpler than γ.

This is pretty much what we did last time – if one's careful, the lemmas we saw can be adapted so that this works & reduce to Dehn twists along curves intersecting the given set of curves in ≤ 2 points. Finish by a case-by-case exploration.

- <u>Hatcher, Thurston 1980</u>: Map_g, $\mathrm{Map}_{g,1}$ are <u>finitely presented</u>

(give an algorithm, used by Harer 1981 to get an explicit presentation – impractical).

<u>New tool</u>: the <u>cut-system complex</u> = 2-dimensional cell complex:

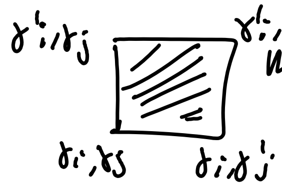
[fix a hyperbolic or flat metric – $g \geq 1$! – & use geodesics to represent curves, so intersection #'s are always minimized in homotopy classes]

- <u>vertices</u> = (unordered) g-tuples $\{\gamma_1, \dots, \gamma_g\}$, $\gamma_i \subset \Sigma_{g,r}$ s.c.c., mutually disjoint, (up to isotopy) s.t. $\Sigma_{g,r} \setminus \cup \gamma_i \simeq \Sigma_{0,2g+r}$

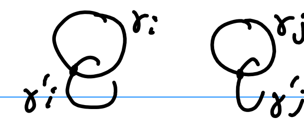
- <u>edges</u>: $\{\gamma_1, \dots, \gamma_i, \dots, \gamma_g\} \longleftrightarrow \{\gamma_1, \dots, \gamma_i', \dots, \gamma_g\}$ where $|\gamma_i \cap \gamma_i'| = 1$.

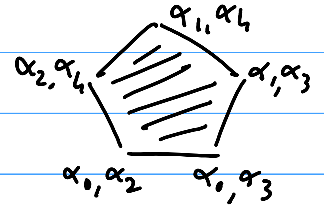
- <u>2-cells</u>: 

w/ vertices = 3 tuples w/ $g-1$ curves in common; differing by $\gamma_i, \gamma_i', \gamma_i''$, $\cap$ each other once

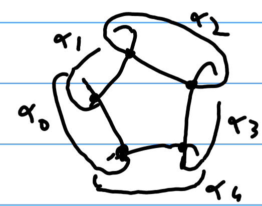


the vertices have $g-2$ curves in common, differ in





$g-2$ curves in common, others form a pentagon



Fact: $\mathrm{Map}(\Sigma)$ acts on this complex $\mathcal{X}$ by cellular maps; the action is transitive on vertices.

Thm (Hatcher–Thurston): $\mathcal{X}$ is connected & simply connected

From there: ① understand the stabilizer of a vertex of $\mathcal{X}$ & give a presentation for it

② derive from this a presentation for $\mathrm{Map}(\Sigma)$.

- Wajnryb 1983: simple presentations of Map_g and $\mathrm{Map}_{g,1}$ using $2g+1$ generators
- for $\mathrm{Map}_{g,r}$, $r \geq 2$: Gervais 1998 – finite presentation. (probably we won't see it)

We'll see the results (not their proofs, though – they're lengthy & technical) but to better motivate these presentations, first need a digression

Digression: branched covers:

Def. a map $f: X \to Y$ is a branched cover if the restriction of f to the complement of codim 2 sets induces an (unramified) covering $f^0: X^0 \to Y^0$

↳ simplicial subsets in the PL category
singular submflds in the smooth category

- ramification set $R \subset X$ = pts near which f is not locally 1-1.
- branch set $\Delta = f(R) \subset Y$ = pts near which f is not locally a covering map.

Ex: $\mathbb{C} \to \mathbb{C}$
$z \mapsto z^p$ is ramified at the origin. ("ramification order" is p).

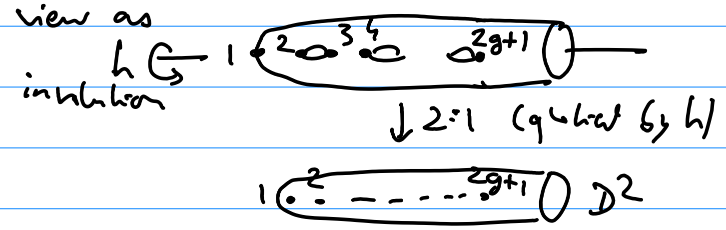
A branched covering of oriented surfaces always looks loc. like this.

Ex: $\mathbb{C} \supset$ disc $D \supset \{q_1, \dots, q_m\} \xleftarrow{pr_x} \Sigma = \{(x,y) \in \mathbb{C}^2 / \ y^2 = (x-q_1)\cdots(x-q_m)\}$

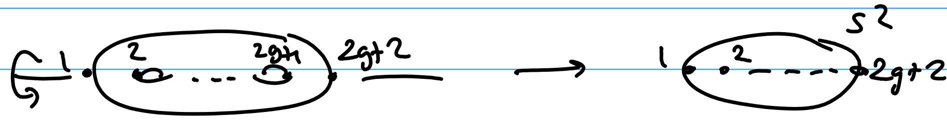
double cover ramified at $q_1 \dots q_n$

smooth; $\begin{cases} m = 2g+1 \Rightarrow \Sigma \simeq \Sigma_{g,1} \\ m = 2g+2 \Rightarrow \Sigma \simeq \Sigma_{g,2} \end{cases}$

view as



extends to $\Sigma_g \xrightarrow[2:1]{} S^2$ branched at $2g+2$ pts



In general an n-fold branched covering of surfaces $\Sigma \xrightarrow[n:1]{f} \Sigma'$ is specified by a homomorphism $\pi_1(\Sigma' \setminus \underbrace{\{q_1, \dots, q_m\}}_{\text{branch set}}) \xrightarrow{\theta} \mathfrak{S}_n$

in $\partial\Sigma'$ if not closed

(implicitly choose a base point, & an identification $f^{-1}(*) \simeq \{1 \dots n\}$)

then given a loop $\gamma \subset \Sigma' - \{q_1, \dots q_n\}$, monodromy of the covering ie. induced deck transformation – follow γ, see who ends where.

E.g.: in above examples of double covers, monodromy around each branch pt is $(1\ 2)$.

The lifting homomorphism:

$f: \Sigma \xrightarrow[n:1]{} \Sigma'$ covering branched at $\{q_1 \dots q_m\} \subset \mathring{\Sigma}'$; assume Σ, Σ' not closed.

Choose a base pt $b \in \partial\Sigma'$ & consider the monodromy homom. $\theta: \pi_1(\Sigma' \setminus \{q_i\}) \to \mathfrak{S}_n$.

& labelling of sheets (ie. $f^{-1}(b) = \{b_1, \dots, b_n\}$)

$\mathrm{Map}_{\{m\}}(\Sigma')$ acts on $\pi_1(\Sigma' \setminus \{q_i\})$; hence also on $\mathrm{Hom}(\pi_1(\Sigma' - \{q_i\}), \mathfrak{S}_n)$

$\hookrightarrow$ homeos fixing $\partial\Sigma'$ and mapping set $\{q_i\}$ to itself.

by composition on the right.

Def: $\phi \in \mathrm{Homeo}^+_{\{n\}}(\Sigma')$ is liftable if $\exists\ \tilde{\phi} \in \mathrm{Homeo}^+(\Sigma)$ s.t. $\begin{array}{ccc} \Sigma & \xrightarrow{\tilde{\phi}} & \Sigma \\ f\downarrow & & \downarrow f \\ \Sigma' & \xrightarrow{\phi} & \Sigma' \end{array}$ commutes

Rmk: if $\tilde{\phi}$ exists then it is unique (assuming Σ' connected)

(indeed: $\forall p \in \Sigma$, know $\tilde{\phi}(p) \in$ fiber above $\phi(f(p))$; which sheet is determined uniquely by looking at lifts of an arc from base pt to $\phi(p)$ in $\Sigma' \setminus \{q_i\}$)

Prop: ϕ is liftable iff induced action on $\pi_1(\Sigma' \setminus \{q_i\})$ satisfies $\theta \circ \phi_* = \theta$.

(in particular, depends only on isotopy class of ϕ; so we get a liftable subgroup of $\mathrm{Map}_{\{m\}}(\Sigma')$ – it depends on θ)

Pf: • Assume ϕ lifts to $\tilde\phi$, and let $\gamma \in \pi_1(\Sigma' \setminus \{q_i\})$. We consider its n lifts $\gamma_1, \dots, \gamma_n$ (arcs in Σ; γ_i starts at i-th lift of base pt, b_i, & ends at $b_{\sigma(i)}$ where $\sigma = \theta(\gamma) \in \mathfrak{S}_n$). Now consider $\hat\gamma = \phi(\gamma)$, and $\hat\gamma_i = \tilde\phi(\gamma_i)$

Since
$$\begin{tikzcd} \Sigma \arrow[r, "\tilde\phi"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \Sigma \arrow[d] \\ \Sigma' \arrow[r, "\phi"'] & \Sigma' \end{tikzcd}$$
, $\hat\gamma_i$ are the lifts of $\hat\gamma$; since $\tilde\phi_{|\partial\Sigma} = \mathrm{Id}$,

end pts of $\hat\gamma_i$ are the same as those of γ_i, namely $\hat\gamma_i$ joins b_i to $b_{\sigma(i)}$; so $\theta(\hat\gamma) = \theta(\gamma)$ ✓

• Conversely, assume $\theta \circ \phi_* = \theta$.
Given $p \in \Sigma$, consider an arc α joining some b_i to p, & whose interior avoids $f^{-1}(\{q_1, \dots q_r\})$. Then p is the end point of the i^{th} lift of the arc $\beta = f(\alpha)$ inside Σ', and we want to define $\tilde\phi(p)$ = the end point of the i^{th} lift of $\phi(\beta)$.
Easy to check: this doesn't depend on the choice of the arc α.
Indeed, if we have another arc α', consider $\alpha \# (-\alpha') = \gamma$, joining b_i to some b_j (j may be $= i$), and $\delta = f(\gamma)$ (a loop in Σ').
If $p \notin f^{-1}(\{q_1, \dots q_r\})$, $\rightarrow$ the i^{th} lift of δ joins b_i to b_j, i.e. $\theta(\delta)$ maps i to j. However $\theta(\phi(\delta)) = \theta(\delta) \Rightarrow$ the i^{th} lift of $\phi(\delta)$ also joins b_i to b_j, which shows our 2 candidates for $\tilde\phi(p)$ coincide
If $p \in f^{-1}(\{q_1, \dots q_r\})$, argue by continuity that this still works

$\square$

• We have a lifting homomorphism associated to the branched cover:
$$\mathrm{Map}_{\{m\}}(\Sigma') \supset \mathrm{Liftable} \longrightarrow \mathrm{Map}(\Sigma)$$

• In the case of double covers of D^2 and S^2 considered above, every mapping class is liftable! Indeed, monodromy @ each branch pt is $(1\,2)$, so θ maps γ to $\underset{\mathrm{Id}}{(1\,2)}$ if total rotation # of γ around q_i's is $\underset{\text{even}}{\text{odd}}$
& this property is clearly preserved under composition with ϕ_* $\forall \phi \in \mathrm{Homeo}^+$