Math 112 Midterm Exam

Tuesday March 12, 2019, 12:00–1:15, in Science Center Hall E.

Rudin's book allowed, no other materials or devices.

YOUR NAME: ______________________

This is a 75-minute in-class exam. No notes or calculators are permitted. Point values (out of 120) are indicated for each problem. Do all the work on these pages. (Use the back if more space is needed)

GRADING	
1. ______	/20
2. ______	/15
3. ______	/25
4. ______	/10
5. ______	/10
6. ______	/25
7. ______	/15
TOTAL	/120

SOLUTIONS

Problem 1. (20 points: 4,4,4,4,4) True or false? (Answers only)

a) The set $\{z = x + iy \in \mathbb{C} : x, y \in \mathbb{Q}\}$ is countable.

[TRUE] **FALSE** (*it is in bijection with* $\mathbb{Q} \times \mathbb{Q}$ *which is countable*)

b) The subset $A = \{1/n,\ n = 1, 2, \dots\}$ of $\mathbb{R}$ is compact.

TRUE **[FALSE]** (*it is not closed:* 0 *is a limit point*)

c) Every bounded infinite subset of $\mathbb{R}$ is contained in a compact set and has a limit point.

[TRUE] **FALSE**

(*bounded* $\Rightarrow$ *contained in a compact interval; sequential compactness* $\Rightarrow \exists$ *limit point*)

d) The subset $\{z \in \mathbb{C},\ |z| \neq 1\}$ of $\mathbb{C}$ is connected.

TRUE **[FALSE]** (*it decomposes as* $\{|z| < 1\} \cup \{|z| > 1\}$)

e) If $a_n \geq 0$ and $a_n \to 0$ then $\sum a_n$ is convergent.

TRUE **[FALSE]** (*counterexample:* $a_n = \frac{1}{n}$)

Problem 2. (15 points: 5,5,5)

Consider the two subsets $\mathbb{Q}$ (the rational numbers) and $\mathbb{Q}^c$ (the irrational numbers) of $\mathbb{R}$ with its usual metric.

a) What are the limit points of $\mathbb{Q}$? What are its interior points? (*No proof needed*)**.**

Every real number is a limit point of $\mathbb{Q}$, since every real number can be approximated by rationals.

$\mathbb{Q}$ has no interior points, since every neighborhood of a rational contains irrationals and hence is not contained in $\mathbb{Q}$.

b) What are the limit points of $\mathbb{Q}^c$? What are its interior points? (*No proof needed*)**.**

Every real number is a limit point of $\mathbb{Q}^c$, since every real number can be approximated by irrationals.

$\mathbb{Q}^c$ has no interior points, since every neighborhood of an irrational contains rationals.

c) Is the following statement correct? If not, what is the error in the proof? (explain briefly)

Claim: *Let A, B be subsets of a metric space. Then every interior point of $E = A \cup B$ is either an interior point of A or an interior point of B.*

Proof: Let x be an interior point of $E = A \cup B$. Then there exists $r > 0$ such that the neighborhood $N_r(x)$ is contained in $A \cup B$. Therefore every point $y \in N_r(x)$ satisfies either $y \in A$ or $y \in B$. In the first case, we conclude that $N_r(x)$ is contained in A, and therefore x is an interior point of A; in the second case, we conclude that $N_r(x)$ is contained in B, and therefore x is an interior point of B.

The result is false (a counterexample is given by $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$). The problem is that $N_r(x)$ may have some of its points belonging to A only and others belonging to B only, so $N_r(x)$ is not necessarily contained in A or in B.

Problem 3. (25 points: 5,5,5,5,5)

If $p = (p_1, p_2)$ and $q = (q_1, q_2)$ are points in $\mathbb{R}^2$, define $d_1(p, q) = |p_1 - q_1| + |p_2 - q_2|$.

a) Prove that d_1 is a metric on $\mathbb{R}^2$.

$d_1(p, q) = |p_1 - q_1| + |p_2 - q_2| \geq 0$ since each term is ≥ 0, and it is equal to zero if and only if $|p_1 - q_1| = |p_2 - q_2| = 0$, i.e. $p_1 = q_1$ and $p_2 = q_2$, i.e. $p = q$.

$d_1(q, p) = |q_1 - p_1| + |q_2 - p_2| = |p_1 - q_1| + |p_2 - q_2| = d_1(p, q)$.

$d_1(p, q) = |p_1 - q_1| + |p_2 - q_2| \leq |p_1 - r_1| + |r_1 - q_1| + |p_2 - r_2| + |r_2 - q_2| = d_1(p, r) + d_1(r, q)$ (using the triangle inequality for the usual distance on $\mathbb{R}$).

So d_1 defines a metric.

b) Prove that $d(p, q) \leq d_1(p, q)$ for all $p, q \in \mathbb{R}^2$, where $d(p, q) = |p - q|$ is the usual distance.

Since $d(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$, we get that $d(p, q)^2 = (p_1 - q_1)^2 + (p_2 - q_2)^2 \leq (p_1 - q_1)^2 + (p_2 - q_2)^2 + 2|p_1 - q_1|\,|p_2 - q_2| = (|p_1 - q_1| + |p_2 - q_2|)^2 = d_1(p, q)^2$. We conclude that $d(p, q) \leq d_1(p, q)$.

c) Prove that, if $p \in \mathbb{R}^2$ and $r > 0$, then the neighborhoods of p for the metrics d_1 and d satisfy the relation $N_r(p, d_1) \subset N_r(p, d)$.

If $x \in N_r(p, d_1)$ then $d_1(p, x) < r$, so by b) we have $d(p, x) \leq d_1(p, x) < r$, so $x \in N_r(p, d)$. Therefore $N_r(p, d_1) \subset N_r(p, d)$.

d) Prove that, if a set E is open in $(\mathbb{R}^2, d)$, then it is open in $(\mathbb{R}^2, d_1)$.

Assume E is open in $(\mathbb{R}^2, d)$, and let $x \in E$. Since x is an interior point of E in $(\mathbb{R}^2, d)$, there exists $r > 0$ uch that $N_r(x, d) \subset E$. Therefore, by c), $N_r(x, d_1) \subset N_r(x, d) \subset E$, so x is an interior point of E in $(\mathbb{R}^2, d_1)$. Since this holds for all $x \in E$, we conclude that E is open in $(\mathbb{R}^2, d_1)$.

e) Prove that, if a set E is closed in $(\mathbb{R}^2, d)$, then it is closed in $(\mathbb{R}^2, d_1)$. (use part d)

If E is closed in $(\mathbb{R}^2, d)$, then E^c is open in $(\mathbb{R}^2, d)$, so by d) E^c is open in $(\mathbb{R}^2, d_1)$, and therefore E is closed in $(\mathbb{R}^2, d_1)$.

Problem 4. (10 points: 5, 5)

Find $\liminf a_n$ and $\limsup a_n$ for each of the following sequences. Are these sequences convergent?

a) $a_n = \sin \frac{n\pi}{4}$**;** **b)** $a_n = \frac{(-1)^n}{n^{3/2}}$**.**

a) the range of the sequence is $\{-1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\}$*, each of these values corresponding to infinitely many terms of the sequence. Therefore* $\liminf a_n = -1$ *and* $\limsup a_n = 1$*; the sequence is not convergent.*

b) $|a_n| = n^{-3/2} \to 0$*, so* $\liminf a_n = \limsup a_n = \lim a_n = 0$*: the sequence is convergent.*

Problem 5. (10 points)

Let $\{p_n\}$ be a sequence in a metric space (X, d), and let $a_n = d(p_n, p_{n+1})$. Assume that the series $\sum a_n$ is convergent. Show that $\{p_n\}$ is a Cauchy sequence.

(*Hint*: use the triangle inequality, and the Cauchy criterion: a series of real or complex numbers converges if and only if $\forall \epsilon > 0\ \exists N$ such that $\forall m \geq n \geq N$, $|\sum_{k=n}^{m} a_k| \leq \epsilon$).

By the triangle inequality, $\forall n < m$*,* $d(p_n, p_m) \leq d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+2}) + \cdots + d(p_{m-1}, p_m) = a_n + a_{n+1} + \cdots + a_{m-1} = \sum_{k=n}^{m-1} a_k$.

Fix $\epsilon > 0$*: since* $\sum a_n$ *converges, by the Cauchy criterion there exists* N *such that* $\forall m \geq n \geq N$*,* $|\sum_{k=n}^{m} a_k| \leq \epsilon$*. Therefore,* $\forall m \geq n \geq N$*,* $d(p_n, p_m) \leq |\sum_{k=n}^{m-1} a_k| \leq \epsilon$*.*

We conclude that $\{p_n\}$ *is a Cauchy sequence.*

Problem 6. (25 points: 10, 15)

Let (X, d) be a metric space and let $f : X \to X$ be a function for which there exists a constant $\alpha > 0$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq \alpha\, d(x, y)$.

a) Prove that, if a sequence x_n converges to a limit x, then $f(x_n)$ converges to $f(x)$.

Fix some $\epsilon > 0$: since $x_n \to x$, there exists N such that $\forall n \geq N$, $d(x_n, x) < \frac{\epsilon}{\alpha}$. By assumption, we have $d(f(x_n), f(x)) \leq \alpha\, d(x_n, x) < \alpha \frac{\epsilon}{\alpha} = \epsilon$. So $\forall \epsilon > 0$ $\exists N$ s.t. $\forall n \geq N$, $d(f(x_n), f(x)) < \epsilon$, i.e. $f(x_n) \to f(x)$.

b) Prove that, if (X, d) is complete and if $0 < \alpha < 1$, then there exists $p \in X$ such that $f(p) = p$.

(*Hint*: Choose $x_1 \in X$ and look at the sequence $x_{n+1} = f(x_n)$; first use Problem 5 to show that $\{x_n\}$ converges to some limit p, then use the result of (a) to show that $f(p) = p$).

Let $x_1 \in X$, and let $x_{n+1} = f(x_n)$. Define $a_n = d(x_n, x_{n+1})$. We have: $a_{n+1} = d(x_{n+1}, x_{n+2}) = d(f(x_n), f(x_{n+1})) \leq \alpha\, d(x_n, x_{n+1}) = \alpha\, a_n$.

We conclude that $\sum a_n$ is convergent, using the comparison test: indeed, by induction on n we have $a_n \leq \alpha^{n-1} a_1$, and since $0 < \alpha < 1$ the geometric series $\sum \alpha^n$ is convergent.

Therefore by Problem 5 the sequence $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some limit $p \in X$. Since $x_n \to p$, by the result of a) we obtain that $f(x_n) \to f(p)$, i.e. $\lim_{n\to\infty} x_{n+1} = f(p)$. However we clearly have $\lim x_{n+1} = p$ (because $\{x_n\}$ and $\{x_{n+1}\}$ have the same limit). Therefore $f(p) = p$.

Problem 7. (15 points: 8, 7)

Assume that $\sum a_n$ is a convergent series and that $a_n \geq 0\ \forall n \in \mathbb{N}$.

a) Prove that $\sum a_n^2$ converges (Hint: use the comparison criterion).

If $\sum a_n$ converges then $a_n \to 0$, so there exists N such that $a_n \leq 1\ \forall n \geq N$. Then $0 \leq a_n^2 \leq a_n$ for $n \geq N$, so by the comparison criterion $\sum a_n^2$ converges.

b) Give an example showing that $\sum n a_n^2$ does not necessarily converge.

(*Easier question for partial credit: show that $\sum n^2 a_n^2$ does not necessarily converge.*)

Let $a_n = \begin{cases} 1/k^2 & \text{if } n = k^4 \\ 0 & \text{otherwise.} \end{cases}$

Then $\sum a_n$ converges since its partial sums are bounded: $\sum_{n=1}^{m^4} a_n = \sum_{k=1}^{m} \frac{1}{k^2}$ and the latter is a convergent series. However $\{n a_n^2\}$ does not converge to 0 (infinitely many of its terms are equal to 1), so $\sum n a_n^2$ does not converge.