Math 112 Spring 2019 – Practice problems for the midterm

The midterm will take place on Tuesday March 12, 12:00-1:15, in Science Center Hall E (usual place and time). It will cover the material seen in lecture up to Tuesday March 5 (most of it included) – specifically, Rudin pages 1-63, minus the appendix to Chapter 1 and the section on perfect sets on p.41-42.

You will be allowed Rudin's book but NO OTHER MATERIALS (no notes, no calculators, no electronics). To prepare for the midterm:

• review the main definitions and theorems from Rudin (those you feel you really ought to know), go over the homework assignments and their solutions (available on the course web page);

• try doing the following practice problems (ideally without using the book).

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1. Let $E \subset \mathbb{R}$ be bounded, nonempty, and suppose $\sup E \notin E$. Show that E is infinite.

2. Let $U, V \subset \mathbb{R}^2$ be open subsets satisfying $\bar{U} = \mathbb{R}^2$, $\bar{V} = \mathbb{R}^2$. Prove that $\overline{U \cap V} = \mathbb{R}^2$.
(Hint: if $E \subset X$ then $\bar{E} = X$ if and only if every nonempty open set in X has non-empty intersection with E).

3. If A and B are compact subsets of X, show that $A \cup B$ is compact.

4. Let $\{x_n\}$ be a sequence in $\mathbb{R}$ satisfying $|x_n| \leq \frac{1}{3^n}$ for each $n \geq 1$. Put $y_n = x_1 + \cdots + x_n$. Prove that the sequence $\{y_n\}$ is convergent.

5. Find all the subsequential limits of each of the following sequences: $a_n = n \sin \frac{n\pi}{4}$; $a_n = 1 - \frac{(-1)^n}{n}$; $a_n = 1 - (-1)^n$. Are these sequences bounded? convergent?

6. Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences in $\mathbb{R}$. Prove that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$. Give an example to show that equality need not hold.

7. Find a countable subset of $\mathbb{R}$ with (a) exactly two limit points; (b) countably many limit points; (c) uncountably many limit points.

8. Let A, B be subsets of a metric space, and denote by A°, B° the sets of interior points of A, B. Prove that $(A \cap B)^\circ = A^\circ \cap B^\circ$.

9. Assume that $\sum a_n$ is a convergent series and that $a_n \geq 0\ \forall n \geq N$. Prove that $\sum \frac{1}{n}\sqrt{a_n}$ converges. (Hint: consider the quantity $(\sqrt{a_n} - \frac{1}{n})^2$, and use the comparison criterion).

10. Give an example of a countable compact subset of $(\mathbb{R}, d)$.

11. True or false?
– if a subset $A \subset \mathbb{R}$ has a least upper bound in $\mathbb{R}$ then it also has a greatest lower bound in $\mathbb{R}$;
– if E is a finite subset of a metric space (X, d) then E is closed in X;
– if K is a compact subset of a metric space (X, d) and $F \subset X$ is closed in X, then $K \cap F$ is closed in X.

12. Let E be an open subset of $\mathbb{R}^2$. Is every point of E a limit point of E? Same question if E is closed.

13. If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2 + s_n}$ $(n = 1, 2, 3, \dots)$, prove that $s_n < 2$ for all n and that $\{s_n\}$ converges, (Hint: show that $\{s_n\}$ is a monotonic sequence).

14. Find $\limsup s_n$ and $\liminf s_n$, where $\{s_n\}$ is the sequence defined by $s_1 = 0$, $s_{2m} = \frac{s_{2m-1}}{2}$, $s_{2m+1} = \frac{1}{2} + s_{2m}$.

15. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_k}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.