Math 112 Homework 9 Solutions

Problem 1.

Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)x^n\,dx = 0$ for all integers $n \geq 0$. By the Weierstrass theorem, there exists a sequence of polynomials $\{P_n\}$ converging to f uniformly over $[0,1]$. We know that $\int_0^1 f(x)P_n(x)\,dx = 0$ for every value of n (because P_n is of the form $P_n(x) = \sum_{k=0}^N c_k x^k$, and each of the terms leads to an integral equal to zero).

Since f and P_n are bounded over $[0,1]$ (they are continuous and $[0,1]$ is compact), and since $P_n \to f$ uniformly, we know that fP_n converges uniformly to f^2. Indeed, letting $M = \sup_{x\in[0,1]} |f(x)|$ we have $\sup_{x\in[0,1]} |f(x)P_n(x) - f(x)^2| \leq M \sup_{x\in[0,1]} |P_n(x) - f(x)| \to 0$ as $n \to +\infty$ (or we can apply the result proved in the previous problem set). By Theorem 7.16, the uniform convergence of fP_n to f^2 implies that $\int_0^1 f(x)^2\,dx = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)\,dx = \lim_{n\to\infty} 0 = 0$.

The function f^2 takes non-negative values, is continuous over $[0,1]$, and $\int_0^1 f^2\,dx = 0$. Therefore $f(x)^2 = 0$ for every $x \in [0,1]$ (see e.g. Chapter 6 Exercise 2 and HW7 solutions). Therefore $f = 0$.

Problem 2.

Since $P_{n+1}(x) = P_n(x) + \frac{1}{2}(x^2 - P_n^2(x))$, we have the following identity: $|x| - P_{n+1}(x) = |x| - P_n(x) - \frac{1}{2}(|x| - P_n(x))(|x| + P_n(x)) = \big(|x| - P_n(x)\big)\big(1 - \frac{1}{2}(|x| + P_n(x))\big)$ (*).

We first prove by induction that $0 \leq P_n(x) \leq |x|$ for every $x \in [-1,1]$. Indeed, the statement is clearly true for $n = 0$ since $P_0 = 0$; and if $x \in [-1,1]$ and $0 \leq P_n(x) \leq |x|$ then we have $0 \leq 1 - |x| \leq 1 - \frac{1}{2}(|x| + P_n(x)) \leq 1 - \frac{1}{2}|x| \leq 1$, so from the identity (*) we get $0 \leq |x| - P_{n+1}(x) \leq |x| - P_n(x)$, i.e. $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$. Therefore, the sequence of functions $\{P_n\}$ is increasing, and $0 \leq P_n(x) \leq |x|$ $\forall x \in [-1,1]$ for every n.

Moreover, for $x \in [-1,1]$ the inequality $0 \leq 1 - \frac{1}{2}(|x| + P_n(x)) \leq 1 - \frac{1}{2}|x|$ and the identity (*) imply that $0 \leq |x| - P_{n+1}(x) \leq (|x| - P_n(x))(1 - \frac{1}{2}|x|)$; therefore, by induction on n we get that, for every integer $n \geq 0$ and for every $x \in [-1,1]$, $0 \leq |x| - P_n(x) \leq (|x| - P_0(x))(1 - \frac{1}{2}|x|)^n = |x|(1 - \frac{1}{2}|x|)^n$.

Now define $f(t) = t(1 - \frac{1}{2}t)^n$ for $t \in [0,1]$. The function f is differentiable and its derivative is $f'(t) = (1 - \frac{1}{2}t)^n - t\,\frac{n}{2}(1 - \frac{1}{2}t)^{n-1} = (1 - \frac{n+1}{2}t)(1 - \frac{1}{2}t)^{n-1}$. Observing that $f'(t)$ has the same sign as $(1 - \frac{n+1}{2}t)$ over the interval $[0,1]$, we get that f is increasing on $[0, \frac{2}{n+1}]$ and decreasing on $[\frac{2}{n+1}, 1]$, i.e. it reaches its maximum for $t = \frac{2}{n+1}$. Therefore, $\forall t \in [0,1]$, $f(t) \leq f(\frac{2}{n+1}) = \frac{2}{n+1}(\frac{n}{n+1})^n < \frac{2}{n+1}$. In particular we conclude that, $\forall x \in [-1,1]$, $0 \leq |x| - P_n(x) \leq f(|x|) < \frac{2}{n+1}$. Since $\frac{2}{n+1}$ does not depend on x and converges to 0 as $n \to \infty$, we conclude that $P_n(x)$ converges to $|x|$ uniformly on the interval $[-1,1]$.

Problem 3.

Recall that every positive integer can be expressed in a unique way as a product of prime numbers. Fixing N, let $p_1, \ldots, p_k$ be all prime numbers less than N, and for $1 \leq j \leq k$ let m_j be the largest positive integer such that $p_j^{m_j} \leq N$. Then every integer between 1 and N can be put in the form $p_1^{r_1} \ldots p_k^{r_k}$ for some integers $r_1, \ldots, r_k$ satisfying $0 \leq r_j \leq m_j$. Therefore

$$\sum_{n=1}^{N} \frac{1}{n} \leq \sum_{r_1=0}^{m_1} \cdots \sum_{r_k=0}^{m_k} \frac{1}{p_1^{r_1} \ldots p_k^{r_k}} = \prod_{j=1}^{k} \Big(1 + \frac{1}{p_j} + \cdots + \frac{1}{p_j^{m_j}}\Big) \leq \prod_{j=1}^{k} \Big(\sum_{r=0}^{\infty} \frac{1}{p_j^r}\Big) = \prod_{j=1}^{k} \frac{1}{1 - \frac{1}{p_j}},$$

where we have bounded the partial sum of the $m_j + 1$ first terms of the geometric series $\sum \frac{1}{p_j^r}$ by the total sum $1/(1 - \frac{1}{p_j})$.

Next we prove the inequality $1/(1-x) \le e^{2x}$ for all $0 \le x \le \frac{1}{2}$. Indeed, let $\phi(x) = (1-x)e^{2x}$, and observe that $\phi'(x) = -e^{2x} + 2(1-x)e^{2x} = (1-2x)e^{2x} \ge 0$ for all $x \in [0, \frac{1}{2}]$. Therefore ϕ is increasing over $[0, \frac{1}{2}]$, and since $\phi(0) = 1$ we conclude that $\forall x \in [0, \frac{1}{2}]$, $\phi(x) \ge 1$, i.e. $1/(1-x) \le \exp(2x)$. Applying this for $x = \frac{1}{p_j}$, we have $1/(1 - \frac{1}{p_j}) \le \exp(\frac{2}{p_j})$. Therefore,

$$\sum_{n=1}^{N} \frac{1}{n} \le \prod_{j=1}^{k} \frac{1}{1-\frac{1}{p_j}} \le \prod_{j=1}^{k} e^{2/p_j} = \exp\Big(\sum_{j=1}^{k} \frac{2}{p_j}\Big), \quad \text{which implies that} \quad \sum_{j=1}^{k} \frac{1}{p_j} \ge \frac{1}{2} \log\Big(\sum_{n=1}^{N} \frac{1}{n}\Big)$$

by taking the logarithm of both sides (recall that log is an increasing function). Next observe that, because the series $\sum \frac{1}{n}$ is divergent, the partial sum $\sum_{n=1}^{N} \frac{1}{n}$ can be made as large as desired by taking N sufficiently large. Since $\lim_{x\to+\infty}(\log x) = +\infty$, we conclude that $\log(\sum_{n=1}^{N} \frac{1}{n})$ can be made as large as desired. Therefore the partial sums of the series $\sum_{p \text{ prime}} \frac{1}{p}$ are unbounded, i.e. the series is divergent.

Problem 4.

Observe that f and g are piecewise affine functions, whose graphs approximate that of the logarithm function: the continuous function f is affine over all intervals $[m, m+1]$ and coincides with log at all integers; g is affine over all intervals $[m - \frac{1}{2}, m + \frac{1}{2})$ and its graph is tangent to that of log at all integers (in fact g is discontinuous at $m + \frac{1}{2}$).

For $x \in [m, m+1]$, define $\phi(x) = \log x - f(x)$: substituting $x = m$ and $x = m+1$ in the definition of f we get $\phi(m) = \phi(m+1) = 0$. The function ϕ is differentiable over $[m, m+1]$, and $\phi'(x) = \frac{1}{x} - f'(x) = \frac{1}{x} + \log m - \log(m+1) = \frac{1}{x} - \log(1 + \frac{1}{m})$. Applying the mean value theorem to ϕ over $[m, m+1]$, there exists $\alpha \in (m, m+1)$ such that $\phi(m+1) - \phi(m) = \phi'(\alpha)$, i.e. $\phi'(\alpha) = \frac{1}{\alpha} - \log(1 + \frac{1}{m}) = 0$. (This can also be checked directly but the argument given here is simpler). Since the function ϕ' is decreasing, we have $\phi' \ge 0$ over $[m, \alpha]$ and $\phi' \le 0$ over $[\alpha, m+1]$, so the function ϕ is increasing over $[m, \alpha]$ and decreasing over $[\alpha, m+1]$. In particular, since $\phi(m) = \phi(m+1) = 0$ we conclude that $\phi(x) \ge 0$, i.e. $f(x) \le \log x$ for every $x \in [m, m+1]$. Since this holds $\forall m \ge 1$, we have $f(x) \le \log x$ over $[1, +\infty)$.

For $x \in [m - \frac{1}{2}, m + \frac{1}{2})$, define $\psi(x) = g(x) - \log x$: one easily checks that $\psi(m) = 0$. The function ψ is differentiable over $[m - \frac{1}{2}, m + \frac{1}{2})$, and $\psi'(x) = g'(x) - \frac{1}{x} = \frac{1}{m} - \frac{1}{x}$, which is negative for $x < m$ and positive for $x > m$. Therefore ψ is decreasing over $[m - \frac{1}{2}, m]$ and increasing over $[m, m + \frac{1}{2})$, and so it reaches its minimum at m. We conclude that $\psi(x) \ge \phi(m) = 0$, i.e. $\log x \le g(x)$ for every $x \in [m - \frac{1}{2}, m + \frac{1}{2})$. Since this holds $\forall m \ge 1$, we have $\log x \le g(x)$ over $[\frac{1}{2}, +\infty)$.

The definition of f gives $\int_m^{m+1} f(x)\,dx = \big[((m+1)x - \frac{1}{2}x^2)\log m + (\frac{1}{2}x^2 - mx)\log(m+1)\big]_m^{m+1} = \frac{1}{2}\log m + \frac{1}{2}\log(m+1)$ (this can be obtained directly geometrically). Therefore, breaking the integral into intervals between consecutive integers we have $\int_1^n f(x)\,dx = (\frac{1}{2}\log 2) + (\frac{1}{2}\log 2 + \frac{1}{2}\log 3) + \cdots + (\frac{1}{2}\log(n-1) + \frac{1}{2}\log n) = \log 2 + \cdots + \log n - \frac{1}{2}\log n = \log(n!) - \frac{1}{2}\log n$.

Similarly: $\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} g(x)\,dx = \big[\frac{1}{2m}x^2 - x + x\log m\big]_{m-1/2}^{m+1/2} = \log m$, so $\int_{\frac{3}{2}}^{n+\frac{1}{2}} g(x)\,dx = \log 2 + \cdots + \log n = \log(n!)$. We can also compute $\int_1^{3/2} g(x)\,dx = (\frac{9}{8} - \frac{3}{2}) - (\frac{1}{2} - 1) = \frac{1}{8}$, and $\int_n^{n+\frac{1}{2}} g(x)\,dx > \frac{1}{2}g(n) = \frac{1}{2}\log n$ (because g is increasing). So $\int_1^n g(x)\,dx < \int_1^{n+\frac{1}{2}} g(x)\,dx - \frac{1}{2}\log n = \log(n!) + \frac{1}{8} - \frac{1}{2}\log n$. We conclude that $\int_1^n f(x)\,dx = \log(n!) - \frac{1}{2}\log n > -\frac{1}{8} + \int_1^n g(x)\,dx$.

Finally, integration by parts (using the functions $\log x$ and x) yields the formula $\int_1^n \log x\,dx = n\log n - \int_1^n \frac{1}{x}x\,dx = n\log n - (n-1)$. Therefore, since $f(x) \le \log x \le g(x)$ $\forall x \in [1, n]$ we have $\int_1^n \log x\,dx > \int_1^n f(x)\,dx > -\frac{1}{8} + \int_1^n g(x)\,dx > -\frac{1}{8} + \int_1^n \log x\,dx$, i.e. $n\log n - (n-1) > \log(n!) - \frac{1}{2}\log n > -\frac{1}{8} + n\log n - (n-1)$. Subtracting $n\log n - n$ from every term we conclude that $1 > \log(n!) - (n + \frac{1}{2})\log n + n > \frac{7}{8}$, i.e. (exponentiating), $e^{7/8} < n!/(n^n\sqrt{n}e^{-n}) < e$.