Math 112 Homework 8 Solutions

Problem 1.

To show $f \notin \mathcal{R}$ for any $[a,b]$, we show that for all partitions P, $U(P,f) = b-a$ but $L(P,f) = 0$, which implies that the lower and upper integrals or f are not equal and hence $f \notin \mathcal{R}$.

Fix a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. (Note: the definition on p. 120 allows $x_0 \le x_1 \le \cdots \le x_n$, but if $x_i = x_{i+1}$ then $\Delta x_i = 0$, so we can discard any point that appears more than once). Each interval $[x_{i-1}, x_i]$ contains a rational number (by density of the rational numbers, Theorem 1.20(b)), so $M_i = 1$ for all i. On the other hand, each interval $[x_{i-1}, x_i]$ also contains irrational numbers, so $m_i = 0$ for all i. This implies that $L(P,f) = \sum_{i=1}^n m_i \,\Delta x_i = 0$ and $U(P,f) = \sum_{i=1}^n M_i \,\Delta x_i = \sum_{i=1}^n (x_i - x_{i-1}) = b-a$, which implies that f is not Riemann-integrable.

Problem 2.

Consider the partition $P_n = \{1, 2, \ldots, n\}$ of the interval $[1,n]$. Since f is monotonically decreasing, $\inf_{[i-1,i]} f = f(i)$ and $\sup_{[i-1,i]} f = f(i-1)$, while $\Delta x_i = 1$ for all i, so we have $L(P_n, f) = \sum_{i=2}^n f(i)$ and $U(P_n, f) = \sum_{i=2}^n f(i-1) = \sum_{i=1}^{n-1} f(i)$. Since $L(P_n,f) \le \int_1^n f\,dx \le U(P_n,f)$, we have the inequalities $\sum_{i=2}^n f(i) \le \int_1^n f\,dx \le \sum_{i=1}^{n-1} f(i)$. By comparison, we get that the integral converges if and only if the series converges.

More precisely, assume that $\sum_{i=1}^\infty f(i)$ converges: then the integrals $I_n = \int_1^n f\,dx \le \sum_{i=1}^{n-1} f(i) \le \sum_{i=1}^\infty f(i)$ are bounded, and so $\int_1^\infty f\,dx$ is convergent (remark that $\{I_n\}$ is a bounded monotonically increasing sequence, and that any integral of the form $\int_1^A f\,dx$ can be bounded between I_n and I_{n+1} for some value of n). Conversely, assume that $\int_1^\infty f\,dx$ converges: then the sums $\sum_{i=2}^n f(i) \le \int_1^n f\,dx \le \int_1^\infty f\,dx$ are bounded, so the series $\sum_{i=2}^\infty f(i)$ is convergent (it is a series of non-negative terms and its partial sums are bounded); adding the single term $f(1)$ to the series, $\sum_{i=1}^\infty f(i)$ is also convergent.

Problem 3.

(a) First observe that, since $\frac{1}{p} + \frac{1}{q} = 1$ and p, q are positive, we must have $\frac{1}{p}, \frac{1}{q} < 1$, i.e. $p, q > 1$. For a fixed value of $v \ge 0$, let $\phi : [0, +\infty) \to \mathbb{R}$ be the function defined by $\phi(u) = \frac{1}{p}u^p - uv$. The function ϕ is differentiable and $\phi'(u) = u^{p-1} - v$. Let $\alpha = v^{1/(p-1)}$: since $p - 1 > 0$, ϕ' is a strictly increasing function, so ϕ' takes negative values over $[0,\alpha)$ and positive values over $(\alpha, +\infty)$. Hence, by the mean value theorem, ϕ is strictly decreasing over the interval $[0,\alpha]$ and strictly increasing over the interval $[\alpha, +\infty)$. In particular, $\forall u \ge 0$ we have $\phi(u) \ge \phi(\alpha)$, with equality if and only if $u = \alpha$. Observe that $\frac{p-1}{p} = 1 - \frac{1}{p} = \frac{1}{q}$; therefore $\phi(\alpha) = \frac{1}{p}\alpha^p - \alpha v = \frac{1}{p}v^{p/(p-1)} - v^{1/(p-1)+1} = -\frac{p-1}{p}v^{p/(p-1)} = -\frac{1}{q}v^q$. We conclude that, for every $u, v \ge 0$, $\frac{1}{p}u^p - uv \ge -\frac{1}{q}v^q$, or equivalenty, $uv \le \frac{1}{p}u^p + \frac{1}{q}v^q$, with equality if and only if $u = v^{1/(p-1)}$, i.e. $u^p = v^{p/(p-1)} = v^q$.

(b) Note that, if $f, g \in \mathcal{R}$, then by Theorems 6.11 and 6.13, f^p, g^q and fg are also integrable. By part (a), we have the inequality $f(x)g(x) \le \frac{1}{p}f(x)^p + \frac{1}{q}f(x)^q$ for every $x \in [a,b]$. Therefore, by Theorem 6.12, we have $\int_a^b fg\,dx \le \int_a^b (\frac{1}{p}f^p + \frac{1}{q}f^q)\,dx = \frac{1}{p}\int_a^b f^p\,dx + \frac{1}{q}\int_a^b g^q\,dx = \frac{1}{p} + \frac{1}{q} = 1$, which is the desired result.

(c) Let $I = (\int_a^b |f|^p\,dx)^{1/p}$ and $J = (\int_a^b |g|^q\,dx)^{1/q}$. Observe that $\int_a^b (\frac{1}{I}|f|)^p\,dx = \frac{1}{I^p}\int_a^b |f|^p\,dx = 1$, and similarly $\int_a^b (\frac{1}{J}|g|)^q\,dx = \frac{1}{J^q}\int_a^b |g|^q\,dx = 1$. Therefore, applying the result of (b) to the functions

$\frac{1}{I}|f|$ and $\frac{1}{J}|g|$, we get $\int_a^b \frac{1}{IJ}|fg|\,dx \leq 1$, or equivalently $\int_a^b |fg|\,dx \leq IJ$. Using Theorem 6.13, which remains true for complex-valued functions (cf. Theorem 6.25), we have $|\int_a^b fg\,dx| \leq \int_a^b |fg|\,dx \leq IJ$, which completes the proof.

Problem 4.

(a) For $n = 0$, the formula becomes $f(x) = f(a) + \int_a^x f'(t)\,dt$, which is the fundamental theorem of calculus (Theorem 6.21).

(b) By comparing the given expressions for $n-1$ and n, one sees that it is sufficient to prove that $R_n(x) = \frac{1}{n!}f^{(n)}(a)\,(x-a)^n + R_{n+1}(x)$. We use integration by parts (Theorem 6.22) with $F(t) = f^{(n)}(t)$ and $G(t) = -\frac{1}{n}(x-t)^n$; we have

$$\begin{aligned} R_n(x) &= \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f^{(n)}(t)\,dt = \frac{1}{(n-1)!}\int_a^x F(t)G'(t)\,dt \\ &= \frac{F(x)G(x) - F(a)G(a)}{(n-1)!} - \frac{1}{(n-1)!}\int_a^x F'(t)G(t)\,dt \\ &= \frac{0 + f^{(n)}(a)(x-a)^n}{n\,(n-1)!} + \frac{1}{n\,(n-1)!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt = \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x). \end{aligned}$$

The result follows by induction on n.

Problem 5.

Assume that $\{f_n\}$ converges uniformly to f and $\{g_n\}$ converges uniformly to g. First observe that f is bounded. Indeed, there exists N such that if $n \geq N$ then $|f_n(x) - f(x)| \leq 1\ \forall x \in E$. Therefore, setting $M = 1 + \sup_{x\in E}|f_N(x)|$, for every $x \in E$ we have $|f(x)| \leq |f_N(x)| + 1 \leq M$. Similarly, g is bounded by a constant M'.

Fix $\epsilon > 0$; decreasing ϵ if necessary we can assume that $\frac{\epsilon}{2M'} < 1$. Since $f_n \to f$ uniformly, there exists N_1 such that $\forall n \geq N_1$, $\forall x \in E$, $|f_n(x) - f(x)| \leq \frac{\epsilon}{2M'}$. Similarly there exists N_2 such that $\forall n \geq N_2$, $\forall x \in E$, $|g_n(x) - g(x)| \leq \frac{\epsilon}{2(M+1)}$. Let $n \geq N = \max(N_1, N_2)$ and $x \in E$: we have

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)(g_n(x) - g(x)) + (f_n(x) - f(x))g(x)| \\ &\leq |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)| \\ &\leq (|f(x)| + \tfrac{\epsilon}{2M'})\tfrac{\epsilon}{2(M+1)} + \tfrac{\epsilon}{2M'}|g(x)| \\ &\leq (M+1)\tfrac{\epsilon}{2(M+1)} + M'\tfrac{\epsilon}{2M'} = \epsilon. \end{aligned}$$

So we have proved that $f_n g_n$ converges uniformly to fg.

The assumption that f_n and g_n are bounded is necessary: consider e.g. $E = (0, +\infty)$, $f_n(x) = x$, and $g_n(x) = \frac{1}{x} + \frac{1}{n}$. It is clear that f_n converges uniformly to $f(x) = x$ and g_n converges uniformly to $g(x) = \frac{1}{x}$. We have $f_n(x)g_n(x) = 1 + \frac{x}{n}$, which converges to 1 as $n \to \infty$ for every value of x. However, $f_n g_n$ does not converge uniformly to 1 over $(0, +\infty)$, in fact $(f_n g_n - 1)$ is not even bounded (consider $x \to \infty$).

Problem 6.

If $x > 0$ then $|\frac{1}{1+n^2x}| \leq \frac{1}{n^2x} = \frac{x^{-1}}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, by the comparison criterion $\sum \frac{1}{1+n^2x}$ converges absolutely for all $x > 0$. If $x = 0$ the series is divergent since $\frac{1}{1+n^2x} = 1$.

Given any constant $a > 0$, we have the inequality $|\frac{1}{1+n^2x}| \leq \frac{1}{n^2x} \leq \frac{a^{-1}}{n^2}$ for all $x \in [a, +\infty)$. Observe that the series $\sum \frac{a^{-1}}{n^2}$ is convergent; therefore, by Theorem 7.10, the series $\sum \frac{1}{1+n^2x}$ converges uniformly on the interval $[a, +\infty)$ for all $a > 0$.

We now consider an interval of the form $(0, a)$: fix an integer N, and let $x \in (0, a)$ be such that $x < \frac{1}{N^2}$. Then $\frac{1}{1+N^2x} > \frac{1}{1+1} = \frac{1}{2}$. This proves that $\frac{1}{1+n^2x}$ does not converge uniformly to 0 over $(0, a)$, which is enough to derive a contradiction to the Cauchy criterion (Theorem 7.8): therefore the series does not converge uniformly on any of the intervals $(0, a)$ for $a > 0$.
Alternatively: letting $f_n(x) = \sum_{k=1}^{n} \frac{1}{1+k^2n}$, we set $A_n = \lim_{t\to 0} f_n(t) = n$. If one assumes the sequence $\{f_n\}$ to converge uniformly over $(0, a)$, by Theorem 7.11 the sequence $\{A_n\}$ must be convergent (and its limit equals $\lim_{t\to 0} f(t)$). This gives a contradiction, so the convergence is not uniform over $(0, a)$.

Since the individual terms in the series are continuous functions and since the convergence is uniform over $[a, +\infty)$, by Theorem 7.12 the function f is continuous over $[a, +\infty)$ for all $a > 0$. Therefore f is continuous at every point of $(0, +\infty)$.

The function f is not bounded: given any integer N, if $x \in (0, \frac{1}{N^2})$ then for every $n \leq N$ we have $\frac{1}{1+n^2x} > \frac{1}{1+(n/N)^2} \geq \frac{1}{2}$, so $\sum_{n=1}^{N} \frac{1}{1+n^2x} > \frac{N}{2}$, and therefore $f(x) > \frac{N}{2}$.