Math 112 Homework 8

Due Thursday April 18, 2019, on Canvas.

You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. *No late homeworks will be accepted.*

Please pay attention to the clarity and precision of your answers. Your solutions to the problems should always consist of carefully written mathematical arguments.

Material covered: Rudin pages 120–135, 143–150 (through Theorem 7.12).

Problem 1. (3 points) Rudin Chapter 6 Problem 4.

Problem 2. (5 points) Rudin Chapter 6 Problem 8.

(Hint: Choose a partition of $[1, n]$ so that $U(P, f)$ and $L(P, f)$ are related to partial sums of the series).

Problem 3. (6 points) Rudin Chapter 6 Problem 10 parts (a)(b)(c), with all Stieltjes integrals ($\int \dots d\alpha$) replaced by Riemann integrals ($\int \dots dx$).

Hint: For part (a), you can e.g. study the behavior of the function defined by $\phi(u) = \frac{1}{p}u^p - uv$ for a fixed value of v.

Problem 4. (5 points)

The following result is a slightly different version of Taylor's theorem, where the error term is expressed as an explicit integral:

Let $I \subset \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a function differentiable $(n + 1)$ times, with all its derivatives continuous. Given a, $x \in I$, let

$$R_{n+1}(x) = \frac{1}{n!} \int_a^x (x - t)^n f^{(n+1)}(t)\, dt.$$

Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + R_{n+1}(x).$$

(a) From which theorem does the case $n = 0$ follow?

(b) Assume by induction that the result is true for $n - 1$. Use integration by parts on $R_n(x)$ to prove the theorem.

Problem 5. (5 points)

If $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions which converge uniformly on some set E, prove that $\{f_n g_n\}$ converges uniformly on E. Show that the result becomes false if one does not require the functions f_n and g_n to be bounded.

Problem 6. (5 points) Rudin Chapter 7 Problem 4. (Assume $x \geq 0$ throughout the problem).