Math 112 Homework 7 Solutions

Problem 1.

Assume that $f'(x) > 0$ for every $x \in (a, b)$, and consider two points such that $x_1 < x_2$ in (a, b): by the mean value theorem, $f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$ for some $c \in (x_1, x_2)$; since $f'(c) > 0$ and $x_2 - x_1 > 0$ we conclude that $f(x_1) < f(x_2)$, i.e. that f is strictly increasing. In particular f is injective (one-to-one), and so there exists a well-defined reciprocal function.

Let $I = f((a, b))$, and consider a point $y \in I$: by definition there exists $x \in (a, b)$ such that $f(x) = y$, and since g is the inverse function of f we have $g(y) = x$. We want to show that g is differentiable at y and that $g'(y) = \frac{1}{f'(x)}$. For this purpose, we consider a sequence $\{y_n\}$ of points of I such that $y_n \neq y$ and $y_n \to y$, and we show that $\frac{g(y_n)-g(y)}{y_n-y}$ converges to $\frac{1}{f'(x)}$. Since $y_n \in I$, there exists $x_n \in (a, b)$ such that $f(x_n) = y_n$, and we have $g(y_n) = x_n$, so that $\frac{g(y_n)-g(y)}{y_n-y} = \frac{x_n-x}{f(x_n)-f(x)}$. In order to conclude by using the differentiability of f at x, we first need to show that $x_n \to x$ (i.e., that $g(y_n) \to g(y)$).

At least two methods can be used to show that g is continuous at y:

1. Let $a', b' \in \mathbb{R}$ be such that $a < a' < x < b' < b$, and consider the restriction $\tilde{f}$ of f to the compact interval $[a', b']$. The function $\tilde{f}$ is continuous and one-to-one (because it is strictly increasing). Therefore, by Theorem 4.17 the inverse mapping $\tilde{g}$ is also continuous. However $\tilde{g}$ is just the restriction of g to the interval $f([a', b'])$, of which y is an interior point (we have $f(a') < f(x) = y < f(b')$ since $a' < x < b'$); therefore g is continuous at y.

2. Observe that, if $y_1 < y_2$, then $g(y_1) < g(y_2)$ (because if we had $g(y_1) \geq g(y_2)$, then since f is increasing we would obtain that $y_1 = f(g(y_1)) \geq f(g(y_2)) = y_2$, contradicting the assumption). Therefore g is also a strictly increasing function. Assume that g is not continuous at y: by Theorem 4.29 we have a simple discontinuity, i.e. $g(y-) < g(y+)$. Let $\alpha = g(y-)$ and $\beta = g(y+)$: we know that $\forall t < y$, $g(t) \leq \alpha$, so since f is increasing we have $t = f(g(t)) \leq f(\alpha)$ for every $t < y$; this implies that $f(\alpha) \geq y$. Similarly, $\forall t > y$, $g(t) \geq \beta$, so $t = f(g(t)) \geq f(\beta)$, and therefore $f(\beta) \leq y$. We conclude that $f(\alpha) \geq f(\beta)$ while $\alpha < \beta$, which contradicts the fact that f is strictly increasing. So g is continuous at y.

Since g is continuous at y and $y_n \to y$, we can now conclude that $x_n = g(y_n) \to g(y) = x$. Therefore, by the definition of $f'(x)$, we have $\frac{f(x_n)-f(x)}{x_n-x} \to f'(x)$, and therefore $\frac{g(y_n)-g(y)}{y_n-y} = \frac{x_n-x}{f(x_n)-f(x)} \to \frac{1}{f'(x)}$.

Problem 2.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}$. Observe that $f(0) = 0$ and $f(1) = 0$ (this latter property follows from the relation between the C_i). We know that f is differentiable; therefore, by the mean value theorem there exists $x \in (0, 1)$ such that $f(1) - f(0) = (1 - 0)f'(x)$, i.e. $f'(x) = 0$. Since $f'(x) = C_0 + C_1 x + \cdots + C_n x^n$, the conclusion follows.

Problem 3.

Fix $x_0 \in [a, b]$, and let $M_0 = \sup\{|f(x)|,\ x \in [a, x_0]\}$ and $M_1 = \sup\{|f'(x)|,\ x \in [a, x_0]\}$. The quantity M_0 is well-defined because f is continuous over the compact set $[a, x_0]$ and hence bounded, and M_1 is also well-defined because $|f'(x)| \leq A|f(x)|\ \forall x$; in fact we clearly have $M_1 \leq AM_0$. Pick a point $x \in [a, x_0]$: since $f(a) = 0$, by the mean value theorem we have $f(x) = f(x) - f(a) = (x - a)f'(c)$ for some $c \in (a, x)$; since $|f'(c)| \leq M_1$ and $0 \leq x - a \leq x_0 - a$, we conclude that $|f(x)| \leq (x_0 - a)M_1 \leq (x_0 - a)AM_0$.

Since the bound $|f(x)| \le (x_0 - a)AM_0$ holds for every $x \in [a, x_0]$, and since $M_0 = \sup\{|f(x)|,\ x \in [a, x_0]\}$, we conclude that $M_0 \le (x_0 - a)AM_0$. Therefore, if $(x_0 - a)A < 1$ we must have $M_0 = 0$, which implies that $f = 0$ on the interval $[a, x_0]$.

Choose elements $a = x_0 < x_1 < \cdots < x_n = b$ such that $x_{i+1} - x_i < \frac{1}{A}$ for every value of i: for example, take n large enough and set $x_i = a + \frac{i}{n}(b-a)$. The above argument applied to $[a, x_1] \subset [a, b]$ shows that $f = 0$ on $[a, x_1]$. We next consider the restriction of f to the smaller interval $[x_1, b]$: since we know from the previous step that $f(x_1) = 0$, by applying the same argument to $[x_1, x_2]$ we show that $f = 0$ on $[x_1, x_2]$. And so on by induction: once we have shown that $f(x_i) = 0$, we can consider the restriction of f to $[x_i, b]$; by the above argument $f = 0$ on the interval $[x_i, x_{i+1}]$, and in particular $f(x_{i+1}) = 0$. After n steps, we conclude that $f = 0$ over the entire interval $[a, b]$.

Problem 4.

Taylor's theorem (Theorem 5.15) with $\alpha = 0$ and $\beta = 1$ gives:

$$1 = f(1) = f(0) + f'(0) + \frac{1}{2}f''(0) + \frac{1}{6}f^{(3)}(s) = \frac{1}{2}f''(0) + \frac{1}{6}f^{(3)}(s) \quad \text{for some } s \in (0,1).$$

Applying again Taylor's theorem with $\alpha = 0$ and $\beta = -1$ gives:

$$0 = f(-1) = f(0) - f'(0) + \frac{1}{2}f''(0) - \frac{1}{6}f^{(3)}(t) = \frac{1}{2}f''(0) - \frac{1}{6}f^{(3)}(t) \quad \text{for some } t \in (-1,0).$$

Subtracting the second identity from the first, we find that there exist $s \in (0,1)$ and $t \in (-1,0)$ such that

$$1 = \frac{1}{6}f^{(3)}(s) + \frac{1}{6}f^{(3)}(t),$$

i.e. $f^{(3)}(s) + f^{(3)}(t) = 6$. We conclude that $f^{(3)}(s) \ge 3$ or $f^{(3)}(t) \ge 3$, which implies the desired conclusion.

Problem 5.

Note that the result is false if one does not assume f to be continuous! (see Chapter 6 Problem 1 for a counterexample).

We argue by contradiction: assume that there exists $x \in [a, b]$ such that $f(x) > 0$. Choose $0 < \epsilon < \frac{1}{2}f(x)$: since f is continuous at x, there exists $\delta > 0$ such that $\forall y \in [a, b]$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. In particular, let $\alpha = x - \frac{1}{2}\delta$ and $\beta = x + \frac{1}{2}\delta$ (if one of these numbers lies outside of $[a, b]$ we set $\alpha = a$ or $\beta = b$ instead). We have $\alpha < \beta$, and $\forall y \in [\alpha, \beta]$, $f(y) \ge f(x) - \epsilon > \frac{1}{2}f(x)$. Now consider the partition $P = \{a, \alpha, \beta, b\}$ of $[a, b]$: since $f \ge 0$, the infimum of f over each sub-interval is non-negative, and moreover $\inf_{[\alpha,\beta]} f \ge \frac{1}{2}f(x)$. Therefore we have $L(P, f) \ge 0 + \frac{1}{2}f(x)(\beta - \alpha) + 0 > 0$. As a consequence, $\int_a^b f\,dx = \underline{\int_a^b} f\,dx \ge L(P, f) > 0$, which is a contradiction. So $f(x) = 0\ \forall x \in [a, b]$.