Math 112 Homework 6 Solutions

Problem 1.

Let $p \in \bar{E}$: by an immediate corollary of Theorem 3.2, there exists a sequence $\{p_n\}$ of elements of E such that $p_n \to p$ (if $p \in E$ we can take $p_n = p$; if p is a limit point of E we can use Theorem 3.2(d)). Since f is continuous, we obtain that $f(p_n) \to f(p)$. Since $p_n \in E$, $f(p_n) \in f(E)$; so $f(p)$ is the limit of a sequence of elements of $f(E)$, and therefore $f(p) \in \overline{f(E)}$. So $f(\bar{E}) \subset \overline{f(E)}$.

An example where $f(\bar{E}) \neq \overline{f(E)}$ is given e.g. by $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \exp(x)$ and $E = \mathbb{R}$: We have $f(\bar{E}) = f(E) = (0, +\infty)$, while $\overline{f(E)} = [0, +\infty)$.

Problem 2.

Direct argument: let p be a limit point of $Z(f)$: there exists a sequence $\{p_n\}$ of elements of $Z(f)$ such that $p_n \to p$. Since f is continuous we have $f(p) = \lim f(p_n)$. However $p_n \in Z(f)$, so $f(p_n) = 0$; therefore $f(p) = 0$, i.e. $p \in Z(f)$. So $Z(f)$ contains its limit points, i.e. it is closed.

Alternate method: by Problem 1, $f(\overline{Z(f)}) \subset \overline{f(Z(f))}$. However, by definition $f(p) = 0\ \forall p \in Z(f)$, so $f(Z(f)) \subset \{0\}$. In particular $f(\overline{Z(f)}) \subset \overline{\{0\}} = \{0\}$, i.e. $\forall p \in \overline{Z(f)}$ we have $f(p) = 0$ and therefore $p \in Z(f)$. So $Z(f)$ is closed.

Problem 3.

Recall that E is dense in X if and only if $\bar{E} = X$. Therefore, by Problem 1 we have $f(X) = f(\bar{E}) \subset \overline{f(E)}$. So every point of $f(X)$ is either an element of $f(E)$ or a limit point of $f(E)$, i.e. $f(E)$ is dense in $f(X)$.

Assume that $g(p) = f(p)\ \forall p \in E$, and consider any point $p \in X$. Since $p \in \bar{E} = X$ there exists a sequence $\{p_n\}$ of points of E such that $p_n \to p$. Since f and g are continuous, we have $f(p) = \lim f(p_n)$ and $g(p) = \lim g(p_n)$. However $p_n \in E$, so $f(p_n) = g(p_n)$; by uniqueness of the limit of a sequence, we conclude that $f(p) = g(p)$.

Problem 4.

Note: your solution is incomplete if it assumes e.g. that E is closed, or an interval (-2 points).

Method 1: taking $\epsilon = 1$ in the definition of uniform continuity, there exists $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < 1$. There exists a positive integer q such that $\frac{1}{q} < \delta$. Moreover, since E is bounded there exists an integer N such that $E \subset (-N, N)$. For every integer $-Nq \leq n < Nq$, let $I_n = [\frac{n}{q}, \frac{n+1}{q})$. If $I_n \cap E \neq \emptyset$, pick some element $x_n \in I_n \cap E$, and let $a_n = |f(x_n)| + 1$; then, given any $y \in I_n \cap E$, we have $|x_n - y| < \frac{1}{q} < \delta$ and therefore $|f(y) - f(x_n)| < 1$, so $|f(y)| < |f(x_n)| + 1 = a_n$; if $I_n \cap E = \emptyset$, let $a_n = 0$. Now, let $A = \sup\{a_n,\ -Nq \leq n < Nq\}$: because $E \subset \bigcup I_n$, given any $y \in E$, there exists n such that $y \in E \cap I_n$, and therefore $|f(y)| < a_n \leq A$. Therefore $f(E) \subset [-A, A]$ is bounded.

Method 2: let $\delta > 0$ be the same as above. By contradiction, assume $f(E)$ is not bounded, and for every positive integer n let p_n be a point of E such that $|f(p_n)| > n$. Since E is bounded, by sequential compactness (Theorem 3.6) the sequence $\{p_n\}$ has a convergent subsequence $\{p_{n_k}\}$: letting $q_k = p_{n_k}$, we have $q_k \to q$ for some $q \in \bar{E}$ (but not necessarily in E, so $f(q)$ is not necessarily defined!) and $|f(q_k)| \to +\infty$. Since the sequence $\{q_k\}$ is convergent, it is a Cauchy sequence; so there exists N such that $\forall m, n \geq N$, $|q_m - q_n| < \delta$. Let $A = |f(q_N)| + 1$: since $|f(q_n)| \to +\infty$, there exists $n \geq N$ such that $|f(q_n)| > A$; in particular we have $|q_n - q_N| < \delta$ and $|f(q_n) - f(q_N)| \geq |f(q_n)| - |f(q_N)| > 1$, which contradicts the definition of δ.

If E is not bounded, a counterexample is given by the function $f(x) = x$ which is uniformly continuous (take $\delta = \epsilon$ in the definition!) but not bounded.

Problem 5.

Define a function $g : I \to \mathbb{R}$ by $g(x) = x - f(x)$. We must prove that there exists $x \in [0,1]$ such that $f(x) = x$, i.e. such that $g(x) = 0$. Since f is continuous, g is also continuous. Moreover, $g(0) = -f(0) \leq 0$ since $f(0) \in [0,1]$, and $g(1) = 1 - f(1) \geq 0$ since $f(1) \in [0,1]$. If $g(0) = 0$ or $g(1) = 0$, then we are done. Otherwise, we have $g(0) < 0$ and $g(1) > 0$: therefore, by Theorem 4.23, there exists $x \in (0,1)$ such that $g(x) = 0$, i.e. $f(x) = x$.

Problem 6.

Let $x \in \mathbb{R}$, and let $\{x_k\}$ be a sequence such that $x_k < x$ $\forall k$ and $x_k \to x$. We first show that $f(x_k) \to 0$. Indeed, fix some $\epsilon > 0$. There exists an integer N such that $\frac{1}{N} < \epsilon$. The only points of $\mathbb{R}$ where f takes values larger than $\frac{1}{N}$ are the rationals of the form $\frac{m}{n}$, where $n \leq N$. Let $S = \{\frac{m}{n} \in (x-1, x) \text{ s.t. } n \leq N\}$: clearly S is finite (there are only finitely many choices for n, and for each of these there are finitely many possible values of m). If S is non-empty, let $y = \sup S$, else let $y = x - 1$: in both cases $y < x$, and by construction f is bounded by $\frac{1}{N}$ at every point of (y, x). Since $x_k \to x$ and $x_k < x$, there exists an integer K such that, if $k \geq K$, then $x_k \in (y, x)$. Therefore, for $k \geq K$ we have $0 \leq f(x_k) \leq \frac{1}{N} < \epsilon$. So we have proved that $f(x_k) \to 0$; since this is true for every sequence $\{x_k\}$ that converges to x from the left, we conclude that $f(x-) = 0$.

A similar argument shows that $f(x+) = 0$ $\forall x \in \mathbb{R}$. In other words, we have $\lim_{t \to x} f(t) = 0$ $\forall x \in \mathbb{R}$ (because the left-hand and right-hand limits are both 0; in fact it is possible to argue directly from both sides at once). So, if x is irrational, then we have $\lim_{t \to x} f(t) = f(x) = 0$, and therefore f is continuous at x. However, if $x = \frac{m}{n}$ is rational, then we have $f(x-) = f(x+) = 0$ while $f(x) = \frac{1}{n}$, so f has a simple discontinuity at x.