Math 112 Homework 5 Solutions

Problem 1.

Since $(\sqrt{a_n} - \frac{1}{n})^2 = a_n + \frac{1}{n^2} - \frac{2}{n}\sqrt{a_n}$, we have

$$0 \le \frac{1}{n}\sqrt{a_n} \le \frac{1}{2}(a_n + \frac{1}{n^2}).$$

The series $\sum a_n$ converges by assumption, and the series $\sum \frac{1}{n^2}$ converges, so by the comparison criterion the series $\sum \frac{1}{n}\sqrt{a_n}$ converges.

Problem 2.

a) $\limsup_{n\to\infty} \sqrt[n]{n^3} = \limsup_{n\to\infty} n^{3/n} = \limsup_{n\to\infty} (n^{1/n})^3 = 1$, so $R = 1$.

b) Note that $\sum\limits_{n=0}^{\infty} \dfrac{2^n}{n!} z^n = \sum\limits_{n=0}^{\infty} \dfrac{(2z)^n}{n!} = e^{2z}$. We know that e^z has $R = \infty$, i.e. $\limsup (n!)^{-1/n} = 0$ (see Example 3.40(b)). Therefore $\limsup (2^n/n!)^{1/n} = 2 \limsup (1/n!)^{1/n} = 0$, and $R = \infty$ again.

c) Observe that, for any $k \in \mathbb{Z}$, $\lim_{n\to\infty} (n^k)^{1/n} = 1$. This is because $\lim n^{1/n} = 1$ (Theorem 3.20(c)), so $\lim_{n\to\infty} n^{k/n} = 1$. Hence, we have $\limsup_{n\to\infty} (2^n/n^2)^{1/n} = 2 \limsup_{n\to\infty} n^{-2/n} = 2$, so $R = 1/2$.

d) Similarly, $\limsup_{n\to\infty} (n^3/3^n)^{1/n} = 1/3$, so $R = 3$.

Problem 3.

a) Observe that, if $a_n \ge 1$, then $\frac{a_n}{1+a_n} \ge \frac{a_n}{a_n+a_n} = \frac{1}{2}$, while if $a_n \le 1$, then $\frac{a_n}{1+a_n} \ge \frac{a_n}{1+1} = \frac{a_n}{2}$. We consider the following alternative.

Case 1: there exist infinitely many values of n such that $a_n > 1$. Then the corresponding values of $\frac{a_n}{1+a_n}$ are all greater than $\frac{1}{2}$, so $\frac{a_n}{1+a_n} \not\to 0$, and therefore $\sum \frac{a_n}{1+a_n}$ diverges.

Case 2: for all but finitely many values of n, we have $a_n \le 1$ and therefore $\frac{a_n}{1+a_n} \ge \frac{a_n}{2}$. Since a finite number of terms cannot affect the behavior of the series, by the comparison criterion we can conclude that, since $\sum \frac{a_n}{2}$ is divergent, the series $\sum \frac{a_n}{1+a_n}$ is also divergent.

(Shorter alternate solution: if $a_n \not\to 0$ then $\frac{a_n}{1+a_n} \not\to 0$ and then $\sum \frac{a_n}{1+a_n}$ diverges. If $a_n \to 0$ then there exists N such that, for $n \ge N$, $a_n \le 1$ and therefore $\frac{a_n}{1+a_n} \ge \frac{a_n}{2}$ $\forall n \ge N$, which gives divergence by the comparison criterion).

b) Since $a_n > 0$, the sequence $\{s_n\}$ is monotonically increasing, and therefore $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \ge \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} = \frac{a_{N+1}+\cdots+a_{N+k}}{s_{N+k}} = \frac{s_{N+k}-s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$.

Observe that the divergence of the series $\sum a_n$ implies that the monotonically increasing sequence s_n is not bounded (by Theorem 3.24). In particular, given any integer N, the constant $2s_N$ is not an upper bound for $\{s_n\}$, i.e. there exists m such that $s_m \ge 2s_N$; obviously $m > N$, so we can write $m = N + k$ for some positive integer k. Therefore, for every integer N, there exists an integer k such that $s_{N+k} \ge 2s_N$, i.e. $1 - \frac{s_N}{s_{N+k}} \ge \frac{1}{2}$. As a consequence, we obtain: $\forall N\ \exists k$ such that $\sum_{n=N+1}^{N+k} \frac{a_n}{s_n} \ge \frac{1}{2}$.

Assume that the series $\sum \frac{a_n}{s_n}$ is convergent: then by the Cauchy criterion (Theorem 3.22) there exists N such that, $\forall m \ge n \ge N$, $|\sum_{k=n}^{m} \frac{a_k}{s_k}| \le \frac{1}{3}$. This contradicts the previously obtained statement; therefore $\sum \frac{a_n}{s_n}$ diverges.

c) Since $s_n = s_{n-1} + a_n \geq s_{n-1}$, we have $\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{a_n}{s_{n-1}s_n} \geq \frac{a_n}{s_n^2}$. Therefore $\sum_{n=1}^N \frac{a_n}{s_n^2} \leq \frac{a_1}{s_1^2} + \sum_{n=2}^N (\frac{1}{s_{n-1}} - \frac{1}{s_n}) = \frac{1}{a_1} + (\frac{1}{s_1} - \frac{1}{s_2}) + \cdots + (\frac{1}{s_{N-1}} - \frac{1}{s_N}) = \frac{2}{a_1} - \frac{1}{s_N} < \frac{2}{a_1}$. So the partial sums of the series of non-negative terms $\sum \frac{a_n}{s_n^2}$ are bounded by the constant $\frac{2}{a_1}$, and therefore by Theorem 3.24 the series $\sum \frac{a_n}{s_n^2}$ is convergent.

d) The series $\sum \frac{a_n}{1+na_n}$ can be either divergent or convergent. Divergent examples are easy to come by (e.g., if $a_n = 1$ then $\frac{a_n}{1+na_n} = \frac{1}{n+1}$ and $\sum \frac{1}{n+1}$ is divergent; if $a_n = \frac{1}{n}$ then $\frac{a_n}{1+na_n} = \frac{1}{2n}$ and $\sum \frac{1}{2n}$ is divergent).

An example where $\sum a_n$ diverges but $\sum \frac{a_n}{1+na_n}$ converges is given by setting $a_n = 1$ if there exists k such that $n = k^2$, and $a_n = \frac{1}{n^2}$ otherwise. Because infinitely many terms are equal to 1 the sequence $\{a_n\}$ does not converge to 0; therefore $\sum a_n$ is divergent. Meanwhile, if $n = k^2$ then $\frac{a_n}{1+na_n} = \frac{1}{k^2+1} < \frac{1}{k^2}$, while in the other case $\frac{a_n}{1+na_n} < a_n = \frac{1}{n^2}$. Therefore the partial sums of the series $\sum \frac{a_n}{1+na_n}$ are bounded: $\sum_{n=1}^N \frac{a_n}{1+na_n} \leq \sum_{k^2 \leq N} \frac{1}{k^2} + \sum_{n=1}^N \frac{1}{n^2} \leq \sum_{k=1}^\infty \frac{1}{k^2} + \sum_{n=1}^\infty \frac{1}{n^2}$ (recall $\sum \frac{1}{n^2}$ is convergent). By Theorem 3.24 we conclude that in this example $\sum \frac{a_n}{1+na_n}$ is convergent.

The case of the series $\sum \frac{a_n}{1+n^2 a_n}$ is easier: it is always convergent, by the comparison criterion, because $0 \leq \frac{a_n}{1+n^2 a_n} \leq \frac{1}{n^2}$.