Math 112 Homework 4 Solutions

Problem 1.

(a) Since $d(x, a) \geq 0 \ \forall a \in A$, we always have $d(x, A) \geq 0$. If $d(x, A) = 0$, then for any $r > 0$ we know that r is not a lower bound of $\{d(x, a),\ a \in A\}$, i.e. $\exists a \in A$ such that $d(x, a) < r$. We conclude that $x \in \bar{A}$. Conversely, if $x \in \bar{A}$, then for any $r > 0$ there exists $a \in A$ such that $d(x, a) < r$, so $d(x, A) \leq d(x, a) < r$. We conclude that $d(x, A)$ cannot be positive; so $d(x, A) = 0$.

(b) For every positive integer n, since $d(x, A) = \inf\{d(x, a),\ a \in A\}$, we know that $d(x, A) + \frac{1}{n}$ is not a lower bound of $\{d(x, a),\ a \in A\}$. So there exists $a_n \in A$ such that $r_n = d(x, a_n) < d(x, A) + \frac{1}{n}$. So $d(x, A) \leq r_n < d(x, A) + \frac{1}{n}$. This implies that $\{r_n\}$ converges to $d(x, A)$. (given $\epsilon > 0$, there exists N such that $\frac{1}{N} < \epsilon$, and for every $n \geq N$ we have $|r_n - d(x, A)| < \frac{1}{n} < \epsilon$).

(c) Assume A is compact: then by Theorem 3.6 we know that the sequence $\{a_n\}$ constructed in (b) has a convergent subsequence $\{a_{n_i}\}$. Let $a \in A$ be the limit of this subsequence. Then we have $d(x, a) \leq d(x, a_{n_i}) + d(a_{n_i}, a)$. Since $d(x, a_{n_i}) \to d(x, A)$ and $d(a_{n_i}, a) \to 0$, the right-hand side converges to $d(x, A)$, and so $d(x, a) \leq d(x, A)$. However $d(x, a) \geq d(x, A)$ by definition of $d(x, A)$, so $d(x, a) = d(x, A)$.

Problem 2.

(a) First we prove that $x_n > \sqrt{\alpha}$ for all n. Indeed, $x_1 > \sqrt{\alpha}$; and, assuming that $x_n > \sqrt{\alpha}$, we have $x_{n+1} - \sqrt{\alpha} = \frac{1}{2}(x_n + \frac{\alpha}{x_n} - 2\sqrt{\alpha}) = \frac{1}{2}(\sqrt{x_n} - \sqrt{\alpha/x_n})^2 > 0$, so $x_{n+1} > \sqrt{\alpha}$. So by induction $x_n > \sqrt{\alpha}$ for all n. As a consequence, for all n we have $\frac{\alpha}{x_n} < \sqrt{\alpha} < x_n$, so $x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}) < \frac{1}{2}(x_n + x_n) = x_n$. So the sequence $\{x_n\}$ decreases monotonically.

Since the sequence $\{x_n\}$ decreases monotonically and admits the lower bound $\sqrt{\alpha}$, by Theorem 3.14 it converges to a certain limit $x \geq \sqrt{\alpha}$. Since $x_n \to x$, we have $\frac{\alpha}{x_n} \to \frac{\alpha}{x}$, so $x_{n+1} \to \frac{1}{2}(x + \frac{\alpha}{x})$. However $x_{n+1} \to x$, so by uniqueness of the limit we have $x = \frac{1}{2}(x + \frac{\alpha}{x})$, which implies that $x = \frac{\alpha}{x}$, i.e. $x = \sqrt{\alpha}$. (This can also be shown directly by estimating $|x_{n+1} - \alpha|$).

(b) $\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2}(x_n + \frac{\alpha}{x_n} - 2\sqrt{\alpha}) = \frac{1}{2x_n}(x_n^2 + \alpha - 2x_n\sqrt{\alpha}) = \frac{1}{2x_n}(x_n - \sqrt{\alpha})^2 = \epsilon_n^2/2x_n$. Since $x_n > \sqrt{\alpha}$, we conclude that $\epsilon_{n+1} < \epsilon_n^2/2\sqrt{\alpha}$.

Setting $\beta = 2\sqrt{\alpha}$, we show by induction on n that $\epsilon_{n+1} < \beta(\epsilon_1/\beta)^{2^n}$. For $n = 1$ we have $\epsilon_2 < \epsilon_1^2/\beta = \beta(\epsilon_1/\beta)^2$. Assume that $\epsilon_n < \beta(\epsilon_1/\beta)^{2^n}$: then $\epsilon_{n+1} < \epsilon_n^2/\beta < \beta(\epsilon_1/\beta)^{2^{n+1}}$. So the inequality holds for all $n = 1, 2, 3, \ldots$.

(c) If $\alpha = 3$ and $x_1 = 2$, then $\epsilon_1 = 2 - \sqrt{3} \simeq 0.268 < 0.3$, and $\beta = 2\sqrt{3} \simeq 3.464 > 3$, so $\epsilon_1/\beta < \frac{0.3}{3} = \frac{1}{10}$. Hence $\epsilon_{n+1} < \beta(\epsilon_1/\beta)^{2^n} < \beta\, 10^{-2^n} < 4 \cdot 10^{-2^n}$. So $\epsilon_5 < 4 \cdot 10^{-16}$, $\epsilon_6 < 4 \cdot 10^{-32}$.

Problem 3.

We show first that $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Indeed, fix $\epsilon > 0$: there exists N such that if $m, n \geq N$ then $d(p_n, p_m) < \frac{\epsilon}{2}$. Similarly there exists N' such that if $m, n \geq N'$ then $d(q_n, q_m) < \frac{\epsilon}{2}$. Let $m, n \geq \max(N, N')$: then $d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n) < \frac{\epsilon}{2} + d(p_m, q_m) + \frac{\epsilon}{2} = d(p_m, q_m) + \epsilon$, and similarly (exchanging m and n) $d(p_m, q_m) < d(p_n, q_n) + \epsilon$. So $|d(p_n, q_n) - d(p_m, q_m)| < \epsilon$ for all $m, n \geq \max(N, N')$. Therefore $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}$; since $\mathbb{R}$ is complete, it converges.

Problem 4.

(a) Let $\{p_n\}$, $\{q_n\}$ and $\{r_n\}$ be Cauchy sequences. The first two properties of an equivalence relation are clearly satisfied: first, $d(p_n, p_n) = 0 \ \forall n$, so $\lim d(p_n, p_n) = 0$, and therefore $\{p_n\} \sim \{p_n\}$. Moreover, if $\{p_n\} \sim \{q_n\}$ then by definition $\lim d(p_n, q_n) = 0$, so $\lim d(q_n, p_n) = 0$ and $\{q_n\} \sim \{p_n\}$. Finally, if $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\}$, observe that $0 \leq d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n)$; since by Problem 4 these sequences all converge, $0 \leq \lim d(p_n, r_n) \leq \lim d(p_n, q_n) + \lim d(q_n, r_n) = 0 + 0 = 0$. Therefore $d(p_n, r_n) \to 0$, and we conclude that $\{p_n\} \sim \{r_n\}$. So $\sim$ is an equivalence relation.

(b) Let $P, Q \in X^*$; let $\{p_n\}$, $\{p'_n\}$ be equivalent sequences representing P, and let $\{q_n\}$, $\{q'_n\}$ be equivalent sequences representing Q. Since $d(p'_n, q'_n) \leq d(p'_n, p_n) + d(p_n, q_n) + d(q_n, q'_n)$, we have $\lim d(p'_n, q'_n) \leq \lim d(p'_n, p_n) + \lim d(p_n, q_n) + \lim d(q_n, q'_n)$; since the first and third term converge to 0, we get $\lim d(p'_n, q'_n) \leq \lim d(p_n, q_n)$. Reversing the roles of p_n and p'_n and of q_n and q'_n, we also have the converse inequality; so $\lim d(p'_n, q'_n) = \lim d(p_n, q_n)$, and therefore $\Delta(P, Q)$ is well-defined.

Next, we prove that Δ is a distance: let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be Cauchy sequences in X, representing elements $P, Q, R \in X^*$. First, since $d(p_n, q_n) \geq 0 \ \forall n$, we get that $\Delta(P, Q) = \lim d(p_n, q_n) \geq 0$. Moreover $\Delta(P, Q) = 0$ if and only if $d(p_n, q_n) \to 0$, i.e. if and only if $\{p_n\}$ and $\{q_n\}$ are equivalent, i.e. if and only if $P = Q$. Next, observe that $\Delta(P, Q) = \lim d(p_n, q_n) = \lim d(q_n, p_n) = \Delta(Q, P)$. Finally we check the triangle inequality: since $d(p_n, q_n) \leq d(p_n, r_n) + d(r_n, q_n)$, we have $\Delta(P, Q) = \lim d(p_n, q_n) \leq \lim d(p_n, r_n) + \lim d(r_n, q_n) = \Delta(P, R) + \Delta(R, Q)$. So Δ is a distance function.

(c) Let $\{P_n\}$ be a Cauchy sequence in (X^*, Δ), and choose a representative $\{p_{nk}\}$ for each P_n. Since $\{p_{nk}\}$ is a Cauchy sequence in X, there exists an integer K_n such that if $k, l \geq K_n$ then $d(p_{nk}, p_{nl}) < \frac{1}{n}$. Let $q_n = p_{nK_n}$.

We first show that $\{q_n\}$ is a Cauchy sequence in X. For this purpose, observe that, for every value of k, $d(q_n, q_m) \leq d(q_n, p_{nk}) + d(p_{nk}, p_{mk}) + d(p_{mk}, q_m)$; in particular, if $k \geq \max(K_n, K_m)$, $d(q_n, p_{nk}) = d(p_{nK_n}, p_{nk}) < \frac{1}{n}$, by definition of K_n, and similarly $d(p_{mk}, q_m) = d(p_{mk}, p_{mK_m}) < \frac{1}{m}$. So for $k \geq \max(K_n, K_m)$ we have $d(q_n, q_m) < \frac{1}{n} + d(p_{nk}, p_{mk}) + \frac{1}{m}$. When $k \to \infty$ the right-hand side converges to $\frac{1}{n} + \Delta(P_n, P_m) + \frac{1}{m}$ by definition of $\Delta(P_n, P_m)$; so we conclude that $d(q_n, q_m) \leq \frac{1}{n} + \Delta(P_n, P_m) + \frac{1}{m}$. Fix a constant $\epsilon > 0$: since $\{P_n\}$ is a Cauchy sequence in (X^*, Δ), there exists N such that $\forall n, m \geq N$, $\Delta(P_n, P_m) < \frac{\epsilon}{2}$. Increasing N if necessary we can also assume that $\frac{1}{N} < \frac{\epsilon}{4}$. We conclude that, if $n, m \geq N$, then $d(q_n, q_m) < \frac{1}{N} + \frac{\epsilon}{2} + \frac{1}{N} < \epsilon$. So $\{q_n\}$ is a Cauchy sequence in (X, d), and we can define $Q \in X^*$ to be its equivalence class.

We now show that $P_n \to Q$. Fix $\epsilon > 0$, and let N be such that $d(q_n, q_m) < \epsilon \ \forall n, m \geq N$ (such an N exists because $\{q_n\}$ is a Cauchy sequence). Let $n \geq N$ and $m \geq \max(N, K_n)$: then $d(p_{nm}, q_m) \leq d(p_{nm}, q_n) + d(q_n, q_m)$. Since $m \geq K_n$, the first term is bounded by $\frac{1}{n}$ (recall $q_n = p_{nK_n}$). Since $m, n \geq N$ the second term is bounded by ϵ. So $d(p_{nm}, q_m) < \frac{1}{n} + \epsilon$. If we keep n fixed and let $m \to \infty$, the left-hand side converges to $\lim_{m\to\infty} d(p_{nm}, q_m) = \Delta(P_n, Q)$. So we conclude that $\Delta(P_n, Q) \leq \frac{1}{n} + \epsilon$ for all $n \geq N$. Increasing N if necessary we can assume that $\frac{1}{N} < \epsilon$; the conclusion becomes: $\forall n \geq N$, $\Delta(P_n, Q) < 2\epsilon$. Since 2ϵ can be chosen as small as desired, we conclude that $P_n \to Q$ (if one insists on getting $\Delta(P_n, Q) < \epsilon$ one can also replace ϵ by $\frac{\epsilon}{2}$ in the preceding sentences).

(d) Let $p, q \in X$, and let $P_p = \phi(p)$, $P_q = \phi(q)$. The constant sequences defined by $p_n = p$ and $q_n = q$ represent P_p and P_q respectively, by definition. We have $d(p_n, q_n) = d(p, q)$, so $\Delta(P_p, P_q) = \lim d(p_n, q_n) = d(p, q)$.

(e) Let $P \in X^*$, and let $\{p_n\}$ be a Cauchy sequence representing P. Fix $\epsilon > 0$: there exists N such that $\forall m, n \geq N$, $d(p_n, p_m) < \epsilon$. Consider the constant sequence $q_n = p_N$, which represents the element $P_{p_N} = \phi(p_N)$ in X^*. For all $n \geq N$ we have $d(p_n, q_n) = d(p_n, p_N) < \epsilon$, so taking the limit as $n \to \infty$ we conclude that $\Delta(P, P_{p_N}) = \lim d(p_n, q_n) \leq \epsilon$. So we have shown that there exist elements of $\phi(X)$ which lie at arbitrarily small distance from P. In other words, every element $P \in X^*$ belongs to $\overline{\phi(X)}$. We conclude that $\phi(X)$ is dense in X^*.

Assume that X is complete: then let $P \in X^*$, and let $\{p_n\}$ be a Cauchy sequence in X representing P. Since X is complete, the sequence $\{p_n\}$ converges to some limit $q \in X$. Consider the constant sequence $q_n = q$ representing $P_q = \phi(q)$: since $p_n \to q$, we have $d(p_n, q_n) = d(p_n, q) \to 0$, so the sequences $\{p_n\}$ and $\{q_n\}$ are equivalent (in the sense of part (a)). Therefore they represent the same element in X^*: we have $P = \phi(q)$. So every element of X^* belongs to $\phi(X)$, and we conclude that $\phi(X) = X^*$.