Math 112 Homework 4

Due Tuesday March 5, 2019, on Canvas.

You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. *No late homeworks will be accepted.*

Please pay attention to the clarity and precision of your answers. Your solutions to the problems should always consist of carefully written mathematical arguments.

Material covered: Rudin pages 47–58.

Problem 1. (5 points)

If A is a subset of a metric space (X, d) and $x \in X$, define $d(x, A) = \inf\{d(x, a),\ a \in A\}$.

a) Show that $d(x, A) = 0$ if and only if $x \in \bar{A}$.
(Hint: recall that $x \in \bar{A}$ if and only if $\forall r > 0\ \exists a \in A$ such that $d(x, a) < r$).

b) Show that there exist elements $a_n \in A$ such that the sequence of real numbers $r_n = d(x, a_n)$ converges to $d(x, A)$.

c) Show that, if A is compact, then there exists $a \in A$ such that $d(x, a) = d(x, A)$.
(Hint: use sequential compactness).

Problem 2. (5 points) Rudin Chapter 3 Problem 16.

Hint: for part (a), first prove that $x_n > \sqrt{\alpha}$ for all n.

Problem 3. (3 points) Rudin Chapter 3 Problem 23.

Problem 4. (15 points: 2,3,5,2,3) Rudin Chapter 3 Problem 24. Remarks and hints:

(a) Definition: a relation $\sim$ is an *equivalence relation* if it has the following properties: (i) $x \sim x$ for all x; (ii) if $x \sim y$ then $y \sim x$; (iii) if $x \sim y$ and $y \sim z$ then $x \sim z$. So you just need to prove that the relation defined by $\{p_n\} \sim \{q_n\} \Leftrightarrow d(p_n, q_n) \to 0$ satisfies these three axioms.

(b) An equivalence relation subdivides a set into *equivalence classes*, where each equivalence class consists of mutually equivalent elements (the equivalence class of x consists of all elements y such that $x \sim y$). So the elements of X^* are equivalence classes of Cauchy sequences, i.e. every $P \in X^*$ can be represented by some Cauchy sequence $\{p_n\}$, but if another Cauchy sequence $\{p'_n\}$ is equivalent to $\{p_n\}$ then they represent the same element in X^*. Therefore, to show that Δ is well-defined one needs to show that $\Delta(P, Q)$ does not depend on the chosen representatives $\{p_n\}$ and $\{q_n\}$ of P and Q. Once this is done, the question also asks to show that Δ satisfies the axioms of a distance function.

(c) To show that every Cauchy sequence in (X^*, Δ) is convergent, you may use the following outline of an argument: (0) let $\{P_n\}$ be a Cauchy sequence in (X^*, Δ), and choose a representative $\{p_{nk}\}$ for each P_n. (1) Define a sequence $\{q_n\}$ consisting of one term of each of the sequences $\{p_{nk}\}$ as follows: fixing a value of n, first show that there exists an integer K_n such that if $k, l \geq K_n$ then $d(p_{nk}, p_{nl}) < \frac{1}{n}$, and define $q_n = p_{nK_n}$. (2) Show that $d(q_n, q_m) \leq \frac{1}{n} + \Delta(P_n, P_m) + \frac{1}{m}$ (use that $d(q_n, q_m) \leq d(q_n, p_{nk}) + d(p_{nk}, p_{mk}) + d(p_{mk}, q_m)$). Conclude that $\{q_n\}$ is a Cauchy sequence in (X, d). (3) Let $Q \in X^*$ be the equivalence class of $\{q_n\}$. Fix $\epsilon > 0$, and let N be such that $d(q_n, q_m) < \epsilon$ for every $n, m \geq N$. Prove that if $n \geq N$ and $m \geq \max(N, K_n)$ then $d(p_{nm}, q_m) < \frac{1}{n} + \epsilon$. Conclude that $\Delta(P_n, Q) \leq \frac{1}{n} + \epsilon$ for all $n \geq N$. (4) Conclude that P_n converges to Q in (X^*, Δ).

(e) To show that $\phi(X)$ is dense (i.e. that $\forall P \in X^*$, $\forall \epsilon > 0$, $\exists \phi(p) = P_p \in \phi(X)$ such that $\Delta(P, P_p) < \epsilon$), consider a sequence $\{p_n\}$ representing P and observe that $\Delta(P, P_{p_n})$ becomes small for n large.