Math 112 Homework 3

Due Tuesday February 26, 2019, on Canvas.

You are encouraged to discuss the homework problems with other students. However, what you hand in should reflect your own understanding of the material. You are NOT allowed to copy solutions from other students or other sources. *No late homeworks will be accepted.*

Please pay attention to the clarity and precision of your answers. Your solutions to the problems should always consist of carefully written mathematical arguments.

Material covered: Rudin pages 36–43.

Problem 1. (3 points) Rudin Chapter 2 Problem 12.

Problem 2. (3 points) Rudin Chapter 2 Problem 15.

Problem 3. (5 points) Rudin Chapter 2 Problem 16.

Problem 4. (5 points)

Let X be a metric space. Show that the union of finitely many compact subsets of X is compact. Is a countable union of compact sets always compact?

Problem 5. (6 points)

The goal of this problem is to show that the Heine-Borel theorem, true for Euclidean spaces, is not true in general. Consider the space

$$\ell^\infty = \{\underline{a} = (a_1, a_2, \dots) \mid a_i \in \mathbb{R},\ \sup_{i\geq 1} |a_i| < \infty\}$$

of all bounded sequences of real numbers, and the function $d(\underline{a}, \underline{b}) = \sup_{i\geq 1} |a_i - b_i|$.

a) Show that (ℓ^∞, d) is a metric space. (In particular you must show that $d(\underline{a}, \underline{b}) < \infty$ for all $\underline{a}, \underline{b}$).

b) Show that the unit ball $B = \{\underline{x} \in \ell^\infty,\ d(\underline{x}, \underline{0}) \leq 1\}$ (where $\underline{0}$ is the constant sequence whose terms are all zero) is closed and bounded.

c) Show that B is not compact. Hint: you might try to show that B is not sequentially compact by constructing an infinite subset of B with no limit point. For example you could consider a sequence of points $\underline{e}_n \in \ell^\infty$ such that $d(\underline{e}_n, \underline{e}_m) = 1$ if $n \neq m$.

Problem 6. (5 points) Rudin Chapter 2 Problem 19.

Hint: For part (c), first prove that A and B are open and use part (b). For part (d), use part (c) to prove that given distinct points x, y in X, and a real number r with $0 < r < d(x, y)$, there exists a point z in X such that $d(x, z) = r$. (Hint: Assume no such z exists and contradict the connectedness of X.) Finally, use the fact that an open interval (or segment) in $\mathbb{R}$ is uncountable.