Math 112 Homework 2 Solutions

Problem 1.

Given an integer N, let E_N be the set of equations of the form $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$, where $n \geq 0$ is an integer, $a_0, \ldots, a_n$ are integers (not all zero), and $n + |a_0| + \cdots + |a_n| = N$. For a given value of n, there exist only finitely many tuples of integers $(a_0, \ldots, a_n)$ such that $|a_0| + \cdots + |a_n| = N - n$ (because each a_i can take at most $2(N-n)+1$ values). Allowing n to vary from 1 to N, we get that E_N is the union of N finite sets, and therefore a finite set.

Next, let $A_N \subset \mathbb{R}$ be the set of solutions to the equations in E_N. Since an equation of degree n has at most n solutions, to each element of E_N correspond at most N elements of A_N; therefore, the finiteness of E_N implies that of A_N.

Observe that each set A_N is contained in the set of algebraic numbers. Conversely, let z be an algebraic number, satisfying an equation $a_0 z^n + a_1 z^{n-1} + \cdots + a_n = 0$, and let $N = n + |a_0| + \cdots + |a_n|$: then $z \in A_N$. Therefore the set of algebraic numbers is exactly $A = \bigcup_{N=1}^{\infty} A_N$. Since A is the union of countably many finite sets A_N, it is at most countable. Moreover it is infinite (since $\mathbb{Z} \subset A$), thus A is countable.

Problem 2.

1) d_1 is not a metric because it violates the triangle inequality: for example, $d_1(0,1) = |1-0|^2 = 1$, $d_1(1,2) = |2-1|^2 = 1$, $d_1(0,2) = |2-0|^2 = 4 > d_1(0,1) + d_1(1,2)$.

2) d_2 is not a metric because it does not satisfy the property that $d(x,y) > 0$ whenever $x \neq y$: for example, $d_2(-1,1) = |1-1| = 0$.

3) d_3 is a metric because it satisfies the three axioms of Definition 2.15:

a) If $x \neq y$, then $|x-y| > 0$, so $d_3(x,y) = \dfrac{|x-y|}{1+|x-y|} > 0$. Moreover $d_3(x,x) = 0$.

b) Clearly $d_3(x,y) = d_3(y,x)$.

c) Let $x, y, z \in \mathbb{R}$, and let $\alpha = |x-y|$, $\beta = |y-z|$, $\gamma = |x-z|$. By the triangle inequality for the usual distance we have $\alpha + \beta \geq \gamma$. We get:

$$d_3(x,y) + d_3(y,z) = \frac{\alpha}{1+\alpha} + \frac{\beta}{1+\beta} \geq \frac{\alpha}{1+\alpha+\beta} + \frac{\beta}{1+\alpha+\beta} = \frac{\alpha+\beta}{1+\alpha+\beta} \geq \frac{\gamma}{1+\gamma} = d_3(x,z),$$

where in the last inequality we have used the fact that the function $t \mapsto \dfrac{t}{1+t}$ is increasing.

Problem 3.

(a) We show that d_∞ satisfies the three axioms of Definition 2.15:

a) If $p \neq q$ then there exists $i \in \{1, \ldots, n\}$ such that $p_i \neq q_i$, so $d_\infty(p,q) \geq |p_i - q_i| > 0$. Moreover $d_\infty(p,p) = 0$ clearly.

b) $d_\infty(q,p) = d_\infty(p,q)$ because $|q_i - p_i| = |p_i - q_i|$ $\forall i$.

c) Let $p, q, r \in \mathbb{R}^n$: then for every $i \in \{1, \ldots, n\}$ we have $|p_i - q_i| \leq |p_i - r_i| + |r_i - q_i| \leq d_\infty(p,r) + d_\infty(r,q)$. So $d_\infty(p,r) + d_\infty(r,q)$ is an upper bound for $\{|p_i - q_i|,\ i = 1, \ldots, n\}$. Therefore $d_\infty(p,r) + d_\infty(r,q) \geq \sup\{|p_i - q_i|\} = d_\infty(p,q)$.

(b) Assume E is open in $(\mathbb{R}^n, d)$, and consider any point $p \in E$. By definition E contains a neighborhood of p for the metric d, i.e. we have $\{q \in \mathbb{R}^n,\ d(p,q) < r\} \subset E$ for some $r > 0$.

Consider a point $q \in \mathbb{R}^n$ such that $d_\infty(p,q) < n^{-1/2}r$: then we have $|p_i - q_i| < n^{-1/2}r$ for all i, so that $d(p,q) = (\sum |p_i - q_i|^2)^{-1/2} < r$, and therefore $q \in E$. It follows that E contains the neighborhood $\{q \in \mathbb{R}^n,\ d_\infty(p,q) < n^{-1/2}r\}$ of p for the metric d_∞. Since E contains a neighborhood of each of its points, it is an open subset of $(\mathbb{R}^n, d_\infty)$.

Conversely, assume E is open in $(\mathbb{R}^n, d_\infty)$, and consider $p \in E$. By definition E contains a neighborhood of the form $\{q \in \mathbb{R}^n,\ d_\infty(p,q) < r\}$ for some $r > 0$. Consider a point $q \in \mathbb{R}^n$ such that $d(p,q) < r$: then $|p_i - q_i| < r$ for all i, so that $d_\infty(p,q) < r$ and therefore $q \in E$. It follows that E contains the neighborhood $\{q \in \mathbb{R}^n,\ d(p,q) < r\}$ of p for the metric d. Since E contains a neighborhood of each of its points, it is an open subset of $(\mathbb{R}^n, d)$.

Problem 4.

We verify the various axioms to show that d is a metric:

– by definition $d(p,q) = 1 > 0$ if $p \neq q$, and $d(p,p) = 0$; moreover $d(p,q) = d(q,p)$.

– let $p,q,r \in X$: if $p = q$ then $d(p,q) = 0 \leq d(p,r) + d(r,q)$. If $p \neq q$ then either $r = p$, in which case $d(p,q) = d(r,q) \leq d(p,r) + d(r,q)$, or $r \neq p$, in which case $d(p,q) = 1 = d(p,r) \leq d(p,r) + d(r,q)$.

Therefore d is a metric. We now show that *every* subset $E \subset X$ is open and closed.

– let $E \subset X$, and let $p \in E$: by definition, we have $\{q \in E,\ d(p,q) < \frac{1}{2}\} = \{p\} \subset E$, so E contains a neighborhood of p. Therefore E is open.

– let $E \subset X$, and let $p \in X$: by definition, the neighborhood $\{q \in E,\ d(p,q) < \frac{1}{2}\} = \{p\}$ does not contain any point $q \neq p$; therefore p is not a limit point of E. So E has no limit points, and is therefore closed (every limit point belongs to E).

(another way to prove that E is closed is to observe that E^c is open – or vice versa).

Problem 5.

Let x be a limit point of E': we want to show that $x \in E'$, i.e. that x is a limit point of E. Fix $r > 0$: since x is a limit point of E', there exists $y \in E'$ such that $y \neq x$ and $d(x,y) < \frac{r}{2}$. However, by definition y is a limit point of E, so there exists $z \in E$ such that $z \neq y$ and $d(y,z) < d(x,y) < \frac{r}{2}$. By the triangle inequality we have $d(x,z) \leq d(x,y) + d(y,z) < \frac{r}{2} + \frac{r}{2} = r$. Moreover $x \neq z$ (because $d(z,y) < d(x,y)$). Therefore every neighborhood of x contains a point of E distinct from x, i.e. x is a limit point of E, or equivalently $x \in E'$. Since every limit point of E' belongs to E', we conclude that E' is closed.

Problem 6.

a) For every i, we have $A_i \subset \bigcup_{i=1}^n A_i = B_n \subset \bar{B}_n$. Therefore $\bar{B}_n$ is a closed set containing A_i, and so by Theorem 2.27(c) in Rudin we get that $\bar{A}_i \subset \bar{B}_n$. Since this holds for every i, we get that $\bigcup_{i=1}^n \bar{A}_i \subset \bar{B}_n$.

Conversely, since the sets $\bar{A}_i$ are closed, by Theorem 2.24(d) the set $\bigcup_{i=1}^n \bar{A}_i$ is closed; moreover, since $\bar{A}_i \supset A_i$ we get that $\bigcup_{i=1}^n \bar{A}_i \supset B_n$. So applying again Theorem 2.27(c) we conclude that $\bigcup_{i=1}^n \bar{A}_i \supset \bar{B}_n$, and therefore $\bigcup_{i=1}^n \bar{A}_i = \bar{B}_n$.

b) Same argument as in a): since $A_i \subset B \subset \bar{B}$, $\bar{B}$ is a closed set containing A_i and therefore $\bar{A}_i \subset \bar{B}$. Since this holds for every i, we get that $\bigcup_{i=1}^\infty \bar{A}_i \subset \bar{B}$.

However the converse inclusion no longer holds in general: for example if $A_i = [\frac{1}{i}, 1] \subset \mathbb{R}$, then $B = \bigcup_{i=1}^\infty A_i = (0,1]$, so $\bar{B} = [0,1]$ while $\bigcup_{i=1}^\infty \bar{A}_i = \bigcup_{i=1}^\infty A_i = (0,1]$.