Math 112 Homework 1 Solutions

Problem 1.

By contradiction: take r rational, x irrational, and assume $y = r + x \in \mathbb{Q}$. Since $y \in \mathbb{Q}$, $r \in \mathbb{Q}$ and $\mathbb{Q}$ is a field, $x = y - r \in \mathbb{Q}$; contradiction, hence y is irrational.

Similarly, take $r \neq 0$ rational, x irrational, and assume $y = rx \in \mathbb{Q}$. Since $y \in \mathbb{Q}$, $r \in \mathbb{Q}$, r is non-zero and $\mathbb{Q}$ is a field, $x = y/r \in \mathbb{Q}$; contradiction, hence y is irrational.

Problem 2.

Since A is bounded below, $-A$ is bounded above (because if x is a lower bound of A, i.e. $x \leq y$ $\forall y \in A$, then $-x \geq -y$ $\forall y \in A$, so $-x$ is an upper bound of $-A$). Since A is not empty, $-A$ is not empty either. Therefore, by the least upper bound property of $\mathbb{R}$, the set $-A$ admits a least upper bound $\alpha = \sup(-A)$. We must show that $\inf(A) = -\alpha$.

First we show that $-\alpha$ is a lower bound of A. Let x be any element of A: then $-x \in -A$, so $-x \leq \alpha$ (α is an upper bound of $-A$). Multiplying by -1 we get $x \geq -\alpha$; since this holds for any $x \in A$, we get that $-\alpha$ is a lower bound of A.

Next we show that $-\alpha$ is the greatest lower bound of A. Let y be any lower bound of A: then $\forall x \in A$, $x \geq y$, so $-x \leq -y$. Since all elements of $-A$ are of the form $-x$ where $x \in A$, we get that $-y$ is an upper bound of $-A$. Therefore, $-y \geq \alpha$ (because α is the least upper bound). Multiplying by -1 again we get $y \leq -\alpha$, so $-\alpha$ is the greatest lower bound of A.

Problem 3.

We must check that the two axioms of an order relation (§1.5 of Rudin) hold:

(i) Let $z = a + bi$, $w = c + di \in \mathbb{C}$. We must show that exactly one of the three properties $z < w$, $z = w$ and $w < z$ holds. There are three cases to consider: if $a < c$, then $z < w$ (while $z \neq w$ and $w \not< z$); if $a > c$, then $w < z$ (while $z \neq w$ and $z \not< w$); the last case is $a = c$. When $a = c$, there are again three subcases: if $b < d$ then $z < w$ (while $z \neq w$ and $w \not< z$); if $b > d$ then $w < z$ (while $z \neq w$ and $z \not< w$); if $b = d$ then $w = z$ (while $z \not< w$ and $w \not< z$).

(ii) Let $z = a + bi$, $w = c + di$, $u = e + fi \in \mathbb{C}$. Assume that $z < w$ and $w < u$. We must show that $z < u$. We know that $a \leq c$ and $c \leq e$, therefore $a \leq e$. If $a < e$ then by definition $z < u$. The remaining case to consider is when $a = e$, where c is also necessarily equal to a and e; then we must have $b < d$ and $d < f$, so $b < f$, and therefore $z < u$.

This ordered set does not have the least-upper-bound property: for example consider $A = \{a + bi,\ a < 0\}$: then $c + di$ is an upper bound of A if and only if $c \geq 0$. However, given any upper bound $w = c + di$ of A, then $w' = c + (d-1)i$ is also an upper bound of A (since $c \geq 0$), and $w' < w$ (since $d - 1 < d$). So there is no least upper bound of A.

Problem 4.

(a) Since $\mathbb{Q}(\sqrt{2})$ is a subset of $\mathbb{R}$, the usual commutativity, associativity and distributivity properties are clearly satisfied. Moreover it is obvious that 0 and 1 belong to $\mathbb{Q}(\sqrt{2})$; therefore it is enough to check that the usual operations are well-defined in $\mathbb{Q}(\sqrt{2})$ (axioms (A1), (A5), (M1), (M5) of Rudin §1.12).

Let $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$ be two elements of $\mathbb{Q}(\sqrt{2})$. Then

$$x + y = (a + c) + (b + d)\sqrt{2}, \qquad xy = (ac + 2bd) + (ad + bc)\sqrt{2}, \qquad -x = (-a) + (-b)\sqrt{2}$$

are clearly elements of $\mathbb{Q}(\sqrt{2})$.

Moreover, if $x \neq 0$, i.e. if a and b are not simultaneously equal to 0, then $a^2 - 2b^2 \neq 0$ because there is no rational number $r \in \mathbb{Q}$ with the property that $r^2 = 2$; therefore

$$x^{-1} = \frac{1}{a + b\sqrt{2}} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2} \in \mathbb{Q}(\sqrt{2}).$$

Therefore $\mathbb{Q}(\sqrt{2})$ with the usual operations is a subfield of $\mathbb{R}$.

(b) By contradiction: assume that $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$, i.e. there exist $a, b \in \mathbb{Q}$ such that $\sqrt{3} = a + b\sqrt{2}$. Then $(a + b\sqrt{2})^2 = (a^2 + 2b^2) + 2ab\sqrt{2}$, so one gets

$$3 - a^2 - 2b^2 = 2ab\sqrt{2}.$$

Since $\sqrt{2} \notin \mathbb{Q}$ the only possibility is that $2ab = 0$, which implies that either $a = 0$ or $b = 0$. If $a = 0$ then one gets $\sqrt{3} = b\sqrt{2}$, i.e. $\sqrt{6} = 2b \in \mathbb{Q}$, which is a contradiction. If $b = 0$ then one gets $\sqrt{3} = a \in \mathbb{Q}$, which is again a contradiction. Therefore $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$.

(We omit the proof that $\sqrt{3}$ and $\sqrt{6}$ are irrational, which is similar to that for $\sqrt{2}$ given in Rudin).

Problem 5.

Recall that $|z|^2 = z\bar{z}$. Then:

$$\begin{aligned}
& |1 + z_1|^2 + |1 + z_2|^2 + \cdots + |1 + z_n|^2 \\
= \; & (1 + z_1)(1 + \bar{z}_1) + (1 + z_2)(1 + \bar{z}_2) + \cdots + (1 + z_n)(1 + \bar{z}_n) \\
= \; & (1 + z_1 + \bar{z}_1 + |z_1|^2) + (1 + z_2 + \bar{z}_2 + |z_2|^2) + \cdots + (1 + z_n + \bar{z}_n + |z_n|^2) \\
= \; & n + (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2) + (z_1 + z_2 + \cdots + z_n) + (\bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n).
\end{aligned}$$

Since $z_1 + z_2 + \cdots + z_n = 0$ one also has that $\bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n = 0$. In addition $|z_1| = |z_2| = \cdots = |z_n| = 1$, so the sum above is equal to $2n$.

Problem 6.

For $\mathbf{x} = 0$ the statement clearly holds (any non-zero $\mathbf{y} \in \mathbb{R}^k$ satisfies $\mathbf{x} \cdot \mathbf{y} = 0$). So we can restrict ourselves to the case where $\mathbf{x} \neq 0$, i.e. $\mathbf{x} = (x_1, \ldots, x_k)$ where at least one of the x_i is non-zero. By permuting the components if necessary, we can assume without loss of generality that $x_1 \neq 0$. Then let $\mathbf{y} = (-x_2, x_1, 0, \ldots, 0) \in \mathbb{R}^k$: we have that $\mathbf{y} \neq 0$ (its second component is non-zero), and $\mathbf{x} \cdot \mathbf{y} = -x_1 x_2 + x_2 x_1 = 0$.

(Or more geometrically: for $\mathbf{x} \neq 0$, the set $\{\mathbf{y} \in \mathbb{R}^k,\ \mathbf{x} \cdot \mathbf{y} = 0\}$ is a hyperplane, which always contains non-zero elements when $k \geq 2$).

For $k = 1$ this is no longer true: if x is non-zero, then the equation $x \cdot y = 0$ admits $y = 0$ as only solution.

Problem 7.

For every positive integer n, let M_n be the set whose elements are all the subsets of the finite set $\{-n, \ldots, n\}$. The set M_n is finite (in fact it has 2^{2n+1} elements). However, every finite subset of $\mathbb{Z}$ is bounded and therefore contained in $\{-n, \ldots, n\}$ for some integer n (of course n depends on the chosen subset). So every element of M belongs to M_n for some n, and therefore $M = \bigcup_{n=1}^{\infty} M_n$. Since it is a countable union of finite sets, M is at most countable; since M is clearly infinite, it is countable.

Alternative solution: for every integer $n \geq 0$, let A_n be the set of all subsets of $\mathbb{Z}$ containing exactly n elements. The set A_0 admits the empty subset as its only element and is therefore finite. If $n \geq 1$, then to an element $\{x_1, \ldots, x_n\}$ of A_n we can associate the element $(x_1, \ldots, x_n)$ of $\mathbb{Z}^n$ (the set of n-tuples of integers), where the x_i's are ordered so that $x_1 < x_2 < \cdots < x_n$. This defines a 1-1 mapping of A_n into $\mathbb{Z}^n$. However $\mathbb{Z}^n$ is countable (see Rudin §2.13), so A_n which is equivalent to an infinite subset of $\mathbb{Z}^n$ is also countable. We conclude that $M = \bigcup_{n=0}^{\infty} A_n$ is also countable.