DIFFERENT NOTIONS OF COMPACTNESS – MATH 112, 2/19/2019

The goal of this handout is to compare different notions related to compactness, and ultimately show that they are all equivalent to compactness.

Recall that, by definition, a metric space K is compact if *every* open cover of K contains a finite subcover: if $K = \bigcup_{\alpha \in A} G_\alpha$ then there exist finitely many $\alpha_1, \dots, \alpha_n \in A$ such that $K = G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$. (Since compactness is an intrinsic property, here we consider K as a standalone metric space, not as a subset of another metric space.)

A slightly weaker condition would be to only consider *countable* open covers of K, i.e. only require the above property when we assume that A is countable (and hence, relabelling the G_α if needed, we can assume A is the set of positive integers). This property is equivalent to several other useful properties.

Theorem 1. *Let K be a metric space. Any of the following properties implies the two others:*

(1) Every countable open cover of K contains a finite subcover.

(2) If F_n is a sequence of nonempty closed subsets of K such that $F_n \supset F_{n+1}$ for all $n \geq 1$, then $\bigcap_{n=1}^\infty F_n$ is not empty.

(3) Every infinite subset of K has a limit point in K.

The third of these properties is called *sequential compactness*.

Exercise 1. *(a) Show that, if K is compact, then K satisfies the first property of the theorem.*

(b) Give examples showing that $\mathbb{R}$ does not satisfy any of the three properties.

(c) Give examples showing that $(0,1) \subset \mathbb{R}$ does not satisfy any of the three properties. (Note that in (2) we want to consider closed subsets relative to *K).*

Solution: (a) This is obvious: if K is compact, then every open cover of K, and in particular every countable open cover of K, admits a finite subcover.

(b) (1) Let $G_n = (n-1, n+1)$ for $n \in \mathbb{Z}$: then $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} G_n$ is a countable open cover. However, any finite subcollection of the G_n only covers a bounded subset of $\mathbb{R}$ and cannot cover $\mathbb{R}$. (In fact, for $n \in \mathbb{Z}$, the point n only belongs to G_n and not to any G_m for $m \neq n$, so any subcover would need to include all of the G_n.)

(2) Let $F_n = [n, \infty)$: these are nonempty closed subsets of $\mathbb{R}$, and $F_n \supset F_{n+1}$ for all n, but $\bigcap_{n=1}^\infty F_n = \emptyset$. (Indeed, given any $x \in \mathbb{R}$, there exists an integer $m > x$, so $x \notin F_m$, and hence $x \notin \bigcap_{n=1}^\infty F_n \subset F_m$.)

(3) The infinite subset $\mathbb{Z} \subset \mathbb{R}$ has no limit points (indeed, each integer $m \in \mathbb{Z}$ has a neighborhood, for instance $N_{1/2}(m)$, which contains no other points of $\mathbb{Z}$; and the non-integer points of $\mathbb{R}$ also have neighborhoods which contain no integers).

(c) (1) Let $G_n = (\frac{1}{n+1}, 1)$ for all $n \geq 1$, then $(0,1) = \bigcup_{n=1}^\infty G_n$ is a countable open cover, but any finite subcollection $G_{n_1} \cup \dots \cup G_{n_k}$ only covers $(\frac{1}{N+1}, 1)$ where $N = \max\{n_1, \dots, n_k\}$.

(2) Let $F_n = (0, \frac{1}{n+1}] \subset (0,1)$. F_n is closed relative to $(0,1)$ (it is the intersection of $(0,1) \subset \mathbb{R}$ with the closed subset $[0, \frac{1}{n+1}]$ of $\mathbb{R}$; or directly: the only potential limit point of F_n which does not lie in F_n would be 0, but in fact 0 is not a limit point of F_n in $(0,1)$ since it is not in $(0,1)$). The intersection $\bigcap_{n=1}^\infty F_n$ is empty.

(3) The infinite subset $\{\frac{1}{n+1},\ n \geq 1\}$ has no limit points in $(0,1)$. (In $\mathbb{R}$ it would have 0 as its only limit point).

Exercise 2. *Prove Theorem 1.*

(1) $\Rightarrow$ (2): Assume every countable open cover of K contains a finite subcover, and let F_n be a sequence of nonempty closed subsets of K such that $F_n \supset F_{n+1}$ for all $n \geq 1$. Assume by contradiction that $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

Let $G_n = F_n^c$ be the complement of F_n. Then G_n is open, and $\bigcup_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} F_n^c = (\bigcap_{n=1}^{\infty} F_n)^c = K$, so $\{G_n\}$ is a countable open cover of K. By assumption there exists a finite subcover, i.e. there exist finitely many integers $n_1, \dots, n_k$ such that $G_{n_1} \cup \dots \cup G_{n_k} = K$. Taking the complement, $F_{n_1} \cap \dots \cap F_{n_k} = (G_{n_1} \cup \dots \cup G_{n_k})^c = \emptyset$. However, $F_{n_1} \cap \dots \cap F_{n_k} = F_{\max(n_1,\dots,n_k)} \neq \emptyset$ by assumption. Contradiction. So in fact $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

(2) $\Rightarrow$ (1): Assume property (2) holds, and consider a countable open cover of K, $K = \bigcup_{n=1}^{\infty} G_n$. For $m \geq 1$, let $U_m = \bigcup_{n=1}^{m} G_m$: then U_m are open subsets, with $U_m \subset U_{m+1}$ for all $m \geq 1$, and $\bigcup_{m=1}^{\infty} U_m = K$.

Let $F_m = U_m^c$, then F_m is closed and $F_m \supset F_{m+1}$ for all $m \geq 1$. If F_m were non-empty for all m, then by our assumption on K it would follow that $\bigcap_{m=1}^{\infty} F_m \neq \emptyset$. However, $\bigcap_{m=1}^{\infty} F_m = (\bigcup_{m=1}^{\infty} U_m)^c = K^c = \emptyset$. So in fact there exists an integer m such that $F_m = \emptyset$, i.e. $U_m = G_1 \cup \dots \cup G_m = K$. Thus $\{G_n\}$ has a finite subcover.

(2) $\Rightarrow$ (3): Assume property (2) holds, and by contradiction assume that K contains an infinite subset E with no limit point. Pick an infinite sequence of distinct points $p_1, p_2, \dots$ in E, and let $F_n = \{p_n, p_{n+1}, \dots\} \subset E$.

Since F_n is contained in E, any limit point of F_n would be a limit point of E, therefore F_n has no limit points, and hence F_n is closed. Moreover, F_n is non-empty, and $F_n \supset F_{n+1}$. However, $\bigcap_{n=1}^{\infty} F_n = \emptyset$, since none of the points p_n belongs to this intersection (as $p_n \notin F_{n+1}$). This contradicts property (2).

(3) $\Rightarrow$ (2): Let F_n be a sequence of non-empty closed subsets of K with $F_n \supset F_{n+1}$. Take $x_n \in F_n$ for each integer n, and let $E = \{x_n,\ n = 1, 2, \dots\}$. If E is finite then one of the x_i belongs to infinitely many F_n. Since $F_1 \supset F_2 \supset \dots$, this implies that x_i belongs to every F_n, and we get that $\bigcap_{n=1}^{\infty} F_n$ is not empty.

Assume now that E is infinite. Since K is sequentially compact, E has a limit point y. We now show that $y \in \bigcap_{n=1}^{\infty} F_n$. For any given integer n, we show y is a limit point of F_n. Indeed, every neighborhood of y contains infinitely many points of E, distinct from y; among them, we can find one which is of the form x_i for $i \geq n$ and therefore belongs to F_n (because $x_i \in F_i \subset F_n$). Since every neighborhood of y contains a point of F_n distinct from y, we get that y is a limit point of F_n; since F_n is closed, this implies that $y \in F_n$. This holds for every n, so we conclude that $y \in \bigcap_{n=1}^{\infty} F_n$, which proves that the intersection is not empty. □

If K is compact, then it satisfies the three properties in Theorem 1; in particular it is sequentially compact. (See also Rudin: Corollary of 2.36, and Theorem 2.37). To show that sequential compactness is in fact *equivalent* to compactness, we now show that every open cover of a sequentially compact set has a *countable* subcover. (Using Theorem 1, there is then a finite subcover, which proves compactness). We first introduce an auxiliary notion.

Definition 1. *A space X is* separable *if it admits an at most countable dense subset.*

For example $\mathbb{R}$ (with the usual distance) is separable ($\mathbb{Q}$ is countable, and it is dense since every real number is a limit of rationals); for the same reason $\mathbb{R}^k$ is separable (consider all points with only rational coordinates).[1]

Theorem 2. *If X is sequentially compact then it is separable.*

Proof. Fix $\delta > 0$, and let $x_1 \in X$. Choose $x_2 \in X$ such that $d(x_1, x_2) \geq \delta$, if possible. Having chosen $x_1, \dots, x_j$, choose x_{j+1} (if possible) such that $d(x_i, x_{j+1}) \geq \delta$ for all $i = 1, \dots, j$. We claim that this process has to stop after a finite number of iterations. Indeed, otherwise we would obtain an infinite sequence of points x_i mutually distant by at least δ; since X is sequentially compact the infinite subset $\{x_i,\ i = 1, 2, \dots\}$ would admit a limit point y, and the neighborhood $N_{\delta/2}(y)$ would contain infinitely many of the x_i's, contradicting the fact that any two of them are distant by at least δ. So after a finite number of iterations we obtain $x_1, \dots, x_j$ such that $N_\delta(x_1) \cup \dots N_\delta(x_j) = X$ (every point of X is at distance $< \delta$ from one of the x_i's).

We now consider this construction for $\delta = \frac{1}{n}$ $(n = 1, 2, \dots)$. For $n = 1$ the construction gives points $x_{11}, \dots, x_{1j_1}$ such that $N_1(x_{11}) \cup \dots \cup N_1(x_{1j_1}) = X$, for $n = 2$ we get $x_{21}, \dots, x_{2j_2}$ such that $N_{1/2}(x_{21} \cup \dots \cup N_{1/2}(x_{2j_2}) = X$, and so on. Let $S = \{x_{ki}, k \geq 1, 1 \leq i \leq j_k\}$: clearly S is at most countable. We claim that S is dense (i.e. $\bar{S} = X$). Indeed, if $x \in X$ and $r > 0$, the neighborhood $N_r(x)$ always contains at least a point of S (choosing n so that $\frac{1}{n} < r$, one of the x_{ni}'s is at distance less than r from x), so every point of X either belongs to S or is a limit point of S, i.e. $\bar{S} = X$. □

Theorem 3. *If X is separable, then every open cover of X admits an at most countable subcover.*

Proof. Let $S = \{p_1, p_2, \dots\}$ be an at most countable subset of X which is dense in X, and let $\{G_\alpha\}_{\alpha \in A}$ be any open cover of X.

Let I be the set of all pairs of positive integers (j, k) for which there exists $\alpha \in A$ such that $N_{1/k}(p_j) \subset G_\alpha$. We claim that $\bigcup_{(j,k)\in I} N_{1/k}(p_j) = X$. Indeed, let $x \in X$: since $\{G_\alpha\}$ covers X, $x \in G_\alpha$ for some $\alpha \in A$. Because G_α is open, it contains a neighborhood $N_r(x)$ for some $r > 0$. Fix k such that $\frac{1}{k} < \frac{r}{2}$. Since S is dense, there exists j such that $d(x, p_j) < \frac{1}{k}$, and we find that $x \in N_{1/k}(p_j) \subset N_r(x) \subset G_\alpha$. So $(j, k) \in I$ and $x \in N_{1/k}(p_j)$.

Next, for each $(j, k) \in I$ we choose $\alpha_{(j,k)} \in A$ such that $N_{1/k}(p_j) \subset G_{\alpha_{j,k}}$. (Such an $\alpha_{(j,k)}$ exists by definition of I; it need not be unique, but we just choose one.)

Then $\bigcup_{(j,k)\in I} G_{\alpha_{j,k}} \supset \bigcup_{(j,k)\in I} N_{1/k}(p_j) = X$, so the $\{G_{\alpha_{j,k}}\}_{(j,k)\in I}$ cover X; since I is at most countable, this subcover consists of at most countably many of the G_α. □

Corollary 1. *Every sequentially compact set is compact.*

Proof. If X is sequentially compact, then by Theorem 2 it is separable, hence by Theorem 3 every open cover of X has an at most countable subcover. However, using the first property in Theorem 1, every countable open cover has a finite subcover, so we conclude that every open cover in X has a finite subcover, i.e. X is compact. □

[1] On the other hand, $\mathbb{R}$ with the distance given by $d(x, y) = 1$ whenever $x \neq y$, $d(x, x) = 0$ is not separable: indeed every subset is closed (HW2 Problem 4), so the only dense subset is $\mathbb{R}$ itself, which is not countable.