LECTURE 1 + PRE-(LECTURE 2) PROBLEM SHEET
INTRODUCTION TO COUNTING ORBITS OF COREGULAR REPRESENTATIONS

Note 1. Don't feel like you have to do these in the order they are written.

Exercise 1 (Two lemmas from lecture)**.** Recall the notion of an `admissible triple` $(\mathscr{O}, I, \delta)$:

- $\mathscr{O}$ is an oriented (i.e. equipped with an isomorphic $\pi\colon \mathscr{O}/\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$) quadratic ring with $\operatorname{Disc}\mathscr{O} \neq 0$;
- $I \subset \mathscr{O} \otimes \mathbb{Q}$ is a `fractional ideal`, i.e. an $\mathscr{O}$-submodule which is $\mathbb{Z}$-free of rank 2;
- $\delta \in (\mathscr{O} \otimes \mathbb{Q})^\times$ satisfies $I^3 \subset \delta\mathscr{O}$ and $N(I)^3 = N(\delta)$.

Recall also the monoid $H(\mathscr{O})$ of equivalence classes of admissible triples (with frist entry $\mathscr{O}$) as well as its subsets (which are groups) $H^{\mathrm{inv}}(\mathscr{O})$ and $H^{\mathrm{red}}(\mathscr{O})$.[1]

Prove the following two lemmas, both stated during the first lecture.

(a) The map $H^{\mathrm{inv}}(\mathscr{O}) \to \mathrm{Cl}(\mathscr{O})$, $[(\mathscr{O}, I, \delta)] \mapsto [I]$ sits in a short exact sequence

$$1 \longrightarrow U^+(\mathscr{O})/3 \longrightarrow H^{\mathrm{inv}}(\mathscr{O}) \longrightarrow \mathrm{Cl}_3(\mathscr{O}) \longrightarrow 1, \text{ where } U^+(\mathscr{O}) := \text{norm } +1 \text{ units in } \mathscr{O}.$$

Above, $U^+(\mathscr{O})/3$ is shorthand for (norm $+1$) units modulo cubes.

(b) Prove that when $\mathscr{O}$ is maximal (equivalently, a Dedekind domain), $H^{\mathrm{red}}(\mathscr{O}) = 1$ is the trivial group.

Exercise 2 (Reducible orbits of $\mathrm{SL}_2 \curvearrowright \mathrm{Sym}^3\mathbb{Z}^2$)**.** Let $\mathcal{I}_3(\mathscr{O})$ denote the group of 3-torsion fractional ideals (not ideal classes), i.e. fractional ideals I such that $I^3 = \mathscr{O}$.

(a) Give an isomorphism $\mathcal{I}_3(\mathscr{O}) \xrightarrow{\sim} H^{\mathrm{red}}(\mathscr{O})$.

(b) Let $\mathscr{O} = \mathbb{Z}[\sqrt{-11}]$, a non-maximal quadratic order, and set $I := (2, (1-\sqrt{-11})/2)$. Prove that $I \neq \mathscr{O}$, but $I \in \mathcal{I}_3(\mathscr{O})$.

Exercise 3 (Reinterpreting $H(\mathscr{O})$)**.** Let K be a number field. Define its 3-Selmer group $\mathrm{Sel}_3(K) := \left\{\delta \in K^\times : \delta\mathscr{O}_K = I^3 \text{ for some fractional ideal } I\right\}/(K^\times)^3$, i.e. the elements $\delta \in K^\times$ whose principal ideal is a cube, modulo cubes in $K^\times$.

(a) When K is quadratic, give an isomorphism $H(\mathscr{O}_K) \xrightarrow{\sim} \mathrm{Sel}_3(K)$.[2]

(b) If you are comfortable with (flat) cohomology, give an isomorphism $\mathrm{Sel}_3(K) \xrightarrow{\sim} \mathrm{H}^1(\mathscr{O}_K, \mu_3)$.

Exercise 4 (Verify the class-number formula)**.** Let $\mathscr{O} = \mathbb{Z}[\sqrt{-2}]$, the ring of integers of $\mathbb{Q}(\sqrt{-2})$. Class-number tables give $\mathrm{Cl}(\mathscr{O}) = \{1\}$, so $|\mathrm{Cl}_3(\mathscr{O})| = 1$ and $|H(\mathscr{O})| = 1$ by a theorem from lecture. Verify the formula by enumerating admissible triples directly: show that the only admissible (I, δ) with $\mathscr{O} = \mathbb{Z}[\sqrt{-2}]$ is, up to equivalence, $(I, \delta) = (\mathscr{O}, 1)$.

Exercise 5 (Quadratic field count)**.** Let $W_{\mathbb{R}} = \left\{f(x,y) = x^2 + axy + by^2 \colon a, b \in \mathbb{R}\right\} \cong \mathbb{R}^2$ be the space of "monic" binary quadratic forms. Let

$$G(\mathbb{R}) = \mathbb{G}_a(\mathbb{R}) \rtimes \mathbb{G}_m(\mathbb{R}) = \left\{\begin{pmatrix} 1 & 0 \\ n & \lambda \end{pmatrix} : n \in \mathbb{R}, \lambda \in \mathbb{R}^\times\right\}$$

act on $W_{\mathbb{R}}$ via

$$(\gamma \cdot f)(x,y) = f((x,y)\cdot\gamma).$$

Above, (x, y) is a row vector and γ multiplied on its right.

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[1] $H^{\mathrm{inv}}(\mathscr{O})$ is where I is invertible, and $H^{\mathrm{red}}(\mathscr{O})$ is where δ is a cube (so, up to equivalence, $\delta = 1$).

[2] Recall that $H(\mathscr{O}_K)$ is the group of (equivalence classes of) admissible triples $(\mathscr{O}, I, \delta)$ where $\mathscr{O} = \mathscr{O}_K$.

- **(a)** Show that, with coordinates (n, λ) on $G(\mathbb{R})$ and (a, b) on $W_{\mathbb{R}}$, the action is given by
$$(n, \lambda) \cdot (a, b) = \left(2n + a\lambda, n^2 + an\lambda + b\lambda^2\right).$$
- **(b)** Show that the discriminant $\mathrm{Disc} = a^2 - 4b$ is invariant under the action of $G(\mathbb{Z})$ (i.e. $n \in \mathbb{Z}$ and $\lambda \in \mathbb{Z}^\times$). What about the action of $G(\mathbb{R})$?
- **(c)** Show that $G(\mathbb{Z})$-orbits on $W_{\mathbb{Z}}$ are in bijection with isomorphism classes of rank-2 $\mathbb{Z}$-algebras, via $(a, b) \mapsto \mathbb{Z}[x]/(x^2 + ax + b)$. Further show that discriminants match.
- **(d)** Show that $G(\mathbb{Z})\backslash W_{\mathbb{Z}}$ has $\{0, 1\} \times \mathbb{Z}$ as a fundamental domain.
- **(e)** Show that the number of (integer) points with $0 < \mathrm{Disc} < X$ (or $-X < \mathrm{Disc} < 0$) is $X/2 + O(1)$.
- **(f)** Use this to deduce that the number of quadratic fields F with $0 < \mathrm{Disc}\, F < X$ (or $-X < \mathrm{Disc}\, F < 0$) is
$$\frac{1}{2\zeta(2)} X + o(X) = \frac{3}{\pi^2} X + o(X).$$
(Hint: the local density for maximality at a prime p is $1 - p^{-2}$).
- **(g)** Give an easier proof of the previous fact by counting fundamental discriminants.

Exercise 6 (Most reducible orbits have $a = 0$)**.** Let $V_{\mathbb{R}} = \left\{ax^3 + 3bx^2y + 3cxy^2 + dy^3 \colon a, b, c, d \in \mathbb{R}\right\}$ be the space of binary cubic forms. Recall that $\mathrm{SL}_2 \curvearrowright V_{\mathbb{R}}$ via $(\gamma \cdot f)(x, y) = f((x, y) \cdot \gamma)$ and that $V_{\mathbb{Z}} \subset V_{\mathbb{R}}$ is the subset of those forms with $a, b, c, d \in \mathbb{Z}$. Fix some constant $C > 0$ and some $f \in V_{\mathbb{R}}$ with coefficients $a(f), b(f), c(f), d(f) \in [-C, C]$ and with $|\mathrm{Disc}\, f| \geq 1$. Show that the number of *integral*, *reducible*[3] binary cubic forms of the form

$$(1) \qquad \begin{pmatrix} 1 & \\ u & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} \cdot \lambda f \in V_{\mathbb{R}} \text{ where } \lambda \in \mathbb{R}^\times_{>0},\ u \in [-1/2, 1/2], \text{ and } t > \sqrt[4]{3}/\sqrt{2}$$

with "leading coefficient" $a \neq 0$ and $|\mathrm{Disc}| < X$ is $O(X^{3/4+\varepsilon})$ (for any $\varepsilon > 0$). Note that there are $\approx cX$ (for some $c > 0$) integral binary cubic forms (with $a \neq 0$) with $|\mathrm{Disc}| < X$, so this says that most of the forms with $a \neq 0$ are irreducible. From another perspective, most reducible orbits have $a = 0$.
Hint: Expand out (1) to write its coefficients in terms of the coefficients of f. Use the fact that $\mathrm{Disc}(\lambda f) = \lambda^4 \mathrm{Disc}(f)$ (whereas $\mathrm{Disc}(\gamma \cdot f) = \mathrm{Disc}(f)$ when $\det \gamma = 1$) to show that $\lambda < X^{1/4}$. Using this and the fact that t is bounded below, you should be able to bound each of the coefficients of (1) in terms of X (e.g. you should show that $a = O(X^{1/4})$). From this, you should deduce that there are $O(X^{3/4})$ such forms with $a \neq 0$ and $d = 0$. To bound the number of reducible forms with $a \neq 0$ and $d \neq 0$, you can use the factorization of (1) into a linear form times a quadratic form and the fact that an integer n will only have $O(n^\varepsilon)$ divisors.

Exercise 7 (The Delone–Faddeev correspondence)**.** Let $V_{\mathbb{Z}} = \{Ax^3 + Bx^2y + Cxy^2 + Dy^3 : A, B, C, D \in \mathbb{Z}\}$, with the twisted $\mathrm{GL}_2(\mathbb{Z})$-action $(\gamma \cdot f)(x, y) = \frac{1}{\det \gamma} f((x, y)\gamma)$. Prove that there is a natural bijection between $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of elements of $V_{\mathbb{Z}}$ and isomorphism classes of cubic rings (rank-3 $\mathbb{Z}$-algebras): given a cubic ring R with a normal basis $\langle 1, \omega, \theta \rangle$ (meaning $\omega\theta \in \mathbb{Z}$), show that the integers A, B, C, D defined by

$$\omega\theta = -AD, \qquad \omega^2 = -AC - B\omega + A\theta, \qquad \theta^2 = -BD - D\omega + C\theta$$

are the coefficients of a well-defined $\mathrm{GL}_2(\mathbb{Z})$-equivalence class, and verify $\mathrm{Disc}(R) = \mathrm{Disc}(f)$ with $\mathrm{Disc}(f) = B^2C^2 - 4AC^3 - 4B^3D - 27A^2D^2 + 18ABCD$.

Exercise 8 (Classical cubic-field count)**.** Combining Problem 7 with the averaging argument of Lecture 1 (adapted to $\mathrm{GL}_2(\mathbb{Z})$ on $V_{\mathbb{Z}}$), and using the following input from local densities:

- at each prime p, the density of $f \in V_{\mathbb{Z}_p}$ for which the associated cubic ring $R(f)$ is maximal at p is $\mu_p(\mathcal{U}_p) = (1 - p^{-2})(1 - p^{-3})$;
- a uniformity estimate: for each p, the number of f with $|\mathrm{Disc}\, f| < X$ and $R(f)$ not maximal at p is $O(X/p^2)$ uniformly in p;

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[3] i.e. forms with a linear factor, so of the form $f(x, y) = (qx + ry)(Ax^2 + Bxy + Cy^2)$

derive the Davenport–Heilbronn theorem on cubic fields:

$$N_3^+(X) = \frac{X}{12\,\zeta(3)} + o(X), \qquad N_3^-(X) = \frac{X}{4\,\zeta(3)} + o(X).$$

(*Hint:* the raw $\mathrm{GL}_2(\mathbb{Z})$-orbit count is $\pi^2 X/(12 n_\pm)$ with $n_+ = 6$, $n_- = 2$; the product of local densities is $\prod_p (1-p^{-2})(1-p^{-3}) = (\zeta(2)\zeta(3))^{-1}$, and $\zeta(2) = \pi^2/6$.)