Talk is based on results obtained together with M. Movshev.

Main goal - analysis of maximally supersymmetric gauge (SUSY YM) theories

Main tools - noncommutative geometry, supergeometry, noncommutative supergeometry, L_∞, A_∞-algebras

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We give a formulation of SUSY YM theories and other YM theories in algebraic terms.

We use it, in particular, to study supersymmetric deformations of these theories.

space $\Longleftrightarrow$ algebra of functions

compact topological space $\Longleftrightarrow$ continuous functions (Gelfand-Naimark)

smooth manifold $\Longleftrightarrow$ smooth functions

formal n-dimensional manifold $\Longleftrightarrow$ formal power series $\mathbb{C}[[x^1,...,x^n]]$

formal (n,m)-dimensional supermanifold $\Longleftrightarrow$ $\mathbb{C}[[x^1,...,x^n]] \otimes \Lambda(\xi^1,\ldots,\xi^m)$

NC space $\Longleftrightarrow$ associative algebra

NC superspace $\Longleftrightarrow$ $\mathbb{Z}_2$-graded assosiative algebra

supermanifold$\Rightarrow$ supercommutative associative algebra

vector field on NC space $\Longleftrightarrow$ derivation of algebra

odd vector field on NC superspace $\Longleftrightarrow$ odd derivation of $\mathbb{Z}_2$-graded algebra

Q-manifold$\Longleftrightarrow$ a supermanifold equipped with odd vector field Q obeying $\{Q,Q\}$=0

NC Q-space $\Longleftrightarrow$ differential $\mathbb{Z}_2$-graded algebra $\Longleftrightarrow$ $\mathbb{Z}_2$-graded algebra equipped with an odd derivation d obeying $d^2 = 0$

Quasiisomorphism of NC Q-spaces $\Longleftrightarrow$ homomorphism of differential $\mathbb{Z}_2$-graded algebras that generates an isomorphism on homology

L_∞ algebra$\Longleftrightarrow$ formal Q-manifold $\Longleftrightarrow$ a differential on the algebra $\mathbb{C}[[x^1,...,x^n]] \otimes \Lambda(\xi^1,\ldots,\xi^m)$

differential Lie algebra $\Rightarrow$ L_∞ algebra

zero locus of vector field $Q$$\Longleftrightarrow$solutions to Maurer-Cartan equations for L_∞ algebra (=usual MC equations in the case of differential Lie algebra)

A_∞ algebra$\Longleftrightarrow$ formal noncommutative Q-manifold $\Longleftrightarrow$ a differential on completion $\mathbb{C} << x^1,...,x^n,\xi^1,\ldots,\xi^m >>$of free non-unital algebra $\mathbb{C} < x^1,...,x^n,\xi^1,\ldots,\xi^m >$ generated by $\mathbb{Z}_2$-graded vector space with even coordinates $x^1,...,x^n$ and odd coordinates $\xi^1,\ldots,\xi^m$

differential associative algebra $\Rightarrow$ A_∞ algebra

A_∞ algebra$A$ $\Rightarrow$$L_\infty$ algebra $L(A)$

We consider an associative algebra $\mathcal{A}$ equipped with an action of Lie algebra L.

One can define a connection (gauge field) on E with respect to L as a set of linear operators $\mathcal{D}_X : E \to E$ depending linearly on $X \in L$ and satisfying an analog of the Leibniz rule

$$\mathcal{D}_X(a \cdot e) = \partial_X(a) \cdot e + a \cdot \mathcal{D}_X e \qquad (1)$$

for any $a \in \mathcal{A}$ and any $e \in E$.

A curvature of connection $\mathcal{D}_X$ is a two-form on L:

$$F_{XY} = [\mathcal{D}_X, \mathcal{D}_Y] - \mathcal{D}_{[X,Y]}\,. \qquad (2)$$

F_{XY} takes values in the algebra $End_{\mathcal{A}}E$ of endomorphisms of the $\mathcal{A}$-module E.

If the module E is projective a trace on the algebra $\mathcal{A}$ induces a trace on $End_{\mathcal{A}}E$ and we can construct Yang-Mills action functional by choosing some invariant metric on L

$$S_{YM}(\mathcal{D}_m) = -\frac{1}{4}\mathrm{Tr}\, F_{mn}F^{mn} \tag{3}$$

where F_{mn} are curvature components in some basis in L.

One can consider gauge fields interacting with other fields (with matter)

Maximally supersymmetric Yang-Mills theory can be constructed in the case when the Lie algebra L acting on $\mathcal{A}$ is ten-dimensional commutative Lie algebra.

Let E denote a projective module over $\mathcal{A}$

To supersymmetrize the action functional (??) we introduce fermionic fields χ^α that take values in $End_{\mathcal{A}}E$ and carry a ten-dimensional spinor index $\alpha = 1, \dots, 16$, i.e.$\chi \in End_{\mathcal{A}}E \times \Pi S$.

We denote by S the space of irreducible spinor representation of $SO(10), \Pi$ stands for parity reversion, Γ^m are Dirac matrices,

Then maximally supersymmetric Yang-Mills action functional can be written as

$$S_{SYM}(\mathcal{D}_m, \chi_\alpha) = -\frac{1}{4}\mathrm{Tr}\, F_{mn}F^{mn} +$$

$$\frac{1}{2}\mathrm{Tr}\, \chi^\alpha \Gamma^m_{\alpha\beta}[\mathcal{D}_m, \chi^\beta]\,. \qquad (4)$$

The action functional (??) is invariant under the supersymmetry transformations

$$\delta_\epsilon(\mathcal{D}_m) = \epsilon^\alpha(\Gamma_m)_{\alpha\beta}\chi^\beta\,,$$
$$\delta_\epsilon(\chi^\alpha) = \frac{1}{2}(\sigma^{mn})^\alpha{}_\beta\epsilon^\beta F_{mn} \tag{5}$$

as well as under a trivial supersymmetry transformations

$$\tilde{\delta}_\epsilon(\mathcal{D}_m) = 0\,, \qquad \tilde{\delta}_\epsilon(\chi^\alpha) = \epsilon^\alpha\,. \tag{6}$$

Here ϵ^α is a constant spinor, i.e. a spinor proportional to the unit endomorphism.

The action (??) is also invariant under gauge transformations parametrized by an endomorphism ϕ:

$$V_\phi(\mathcal{D}_m) = [\mathcal{D}_m, \phi]\,, \qquad V_\phi(\chi^\alpha) = [\chi^\alpha, \phi]\,.$$

The most important cases:

10D SUSY YM theory

$\mathcal{A}$ =**algebra of functions on** R^{10}

Its reductions to four, one and zero dimensions:

N=4 four-dimensional SUSY YM theory

$\mathcal{A}$ =**algebra of functions on** R^4

BFSS model

$\mathcal{A}$ =**algebra of functions on** R^1 **;gauge fields are matrix-valued functions on a line**

IKKT model

$\mathcal{A} = \mathbb{C}$; **gauge fields are matrices**

SUSY YM theory on NC torus

$\mathcal{A}$= **algebra of functions on torus with star-product=** C^***algebra with unitary generators** U_k **and relations** $U_k U_l = e^{i\theta_{kl}} U_l U_k$

Applications of NC geometry to string/M-theory

SUSY YM theory on NC torus can be obtained from BFSS or IKKT model by means of compactification (Connes- Douglas -Schw)

Morita equivalence of NC tori $\Rightarrow$ T-duality (Rieffel-Schw,Schw)

Noncommutative instantons (Nekrasov-Schw)

BPS states in NC SUSY YM (Konechny-Schw)

Background independence (Pioline-Schw, Seiberg-Witten)

Algebraic formulation of SUSY YM and more general YM theories in BV formalism in terms of differential associative algebras and A_∞-algebras

Classical system in Batalin-Vilkovisky (BV) formalism $\Longleftrightarrow$ solution to classical master eqution $\{S, S\}$=0 on odd symplectic manifold $\Longleftrightarrow$ odd symplectic Q-manifold $\Rightarrow L_\infty$ algebra with odd inner product

A_∞**-algebra** $A \Rightarrow A_\infty$**-algebra** $A \otimes Mat_n \Rightarrow L_\infty$ **algebra** $L(A \otimes Mat_n)$

A_∞**-algebra** A **with inner product** $\Rightarrow L_\infty$ **algebra** $L(A \otimes Mat_n)$ **with inner product** $\Rightarrow$ **action functional in BV-formalism**

Gauge theories can be obtained this way from appropriate A_∞-algebras

SUSY YM theory can be obtained from differential associative algebra with inner product $\Rightarrow$ BV-action functional can be written in Chern-Simons form

IKKT theory in BV formalism

pure spinor $u \Longleftrightarrow u\Gamma^m u = 0$

space of pure spinors $OGr(10,5)$ **= total space of line bundle over** $SO(10)/U(5)$

Berkovits algebra B**= algebra of polynomial functions** $p(u,\theta)$ **with differential** $d = u\frac{d}{d\theta}$

Here u **is a pure spinor and** $\theta \in \Pi S$ **is an odd spinor**

Koszul dual to quadratic algebra B is quasi-isomorphic to algebra SYM

Definition of algebra SYM

$$SYM =$$

$$= \mathbb{C} < A_1, \ldots, A_{10}, \chi^1, \ldots, \chi^{16} > / I(YM_k, Dirac_\alpha)$$

$$YM_k = [A_i, [A_i, A_k]] - \frac{1}{2}\Gamma^k_{\alpha\beta}\{\chi^\alpha, \chi^\beta\} \qquad (7)$$

$$Dirac_\alpha = -\Gamma^i_{\alpha\beta}[A_i, \chi^\beta] \qquad (8)$$

Representations of SYM $\Longleftrightarrow$ solutions to SUSY YM equations of motion

Infinitesimal deformations of A_∞ algebra$A$ $\Longleftrightarrow$ Hochschild cohomology $HH(A, A)$

Infinitesimal deformations of A_∞ algebra with invariant inner product $\Longleftrightarrow$ cyclic cohomology (Penkava-Schw)

Equivariant generalizations of these statements are used to analyze SUSY deformations of SUSY YM