CHARACTER SHEAVES AND GENERALIZATIONS

G. Lusztig

Dedicated to I. M. Gelfand on the occasion of his 90th birthday

1. Let $\mathbf{k}$ be an algebraic closure of a finite field $\mathbf{F}_q$. Let $G = GL_n(\mathbf{k})$. The group $G(\mathbf{F}_q) = GL_n(\mathbf{F}_q)$ can be regarded as the fixed point set of the Frobenius map $F : G \to G, (g_{ij}) \mapsto (g_{ij}^q)$. Let $\bar{\mathbf{Q}}_l$ be an algebraic closure of the field of l-adic numbers, where l is a prime number invertible in $\mathbf{k}$. The characters of irreducible representations of $G(\mathbf{F}_q)$ over an algebraically closed field of characteristic 0, which we take to be $\bar{\mathbf{Q}}_l$, have been determined explicitly by J.A.Green [G]. The theory of character sheaves [L2] tries to produce some geometric objects over G from which the irreducible characters of $G(\mathbf{F}_q)$ can be deduced for any q. This allows us to unify the representation theories of $G(\mathbf{F}_q)$ for various q. The geometric objects needed in the theory are provided by intersection cohomology.

Let X be an algebraic variety over $\mathbf{k}$, let X_0 be a locally closed irreducible, smooth subvariety of X and let $\mathcal{E}$ be a local system over X_0 (we say "local system" instead of "$\bar{\mathbf{Q}}_l$-local system"). Deligne, Goresky and MacPherson attach to this datum a canonical object $IC(\bar{X}_0, \mathcal{E})$ (intersection cohomology complex) in the derived category $\mathcal{D}(X)$ of $\bar{\mathbf{Q}}_l$-sheaves on X; this is a complex of sheaves which extends $\mathcal{E}$ to X (by 0 outside the closure $\bar{X}_0$ of X_0) in the most economical possible way so that local Poicaré duality is satisfied. We say that $IC(\bar{X}_0, \mathcal{E})$ is irreducible if $\mathcal{E}$ is irreducible.

Now take $X = G$ and take $X_0 = G_{rs}$ to be the set of regular semisimple elements in G. Let T be the group of diagonal matrices in G. For any integer $m \geq 1$ invertible in $\mathbf{k}$ we have an unramified $n!m^n$-fold covering

$$\pi_m : \{(g, t, xT) \in G_{rs} \times T \times G/T; x^{-1}gx = t^m\} \to G_{rs}, \quad (g, t, xT) \mapsto g.$$

An irreducible local system $\mathcal{E}$ on G_{rs} is said to be admissible if it is a direct summand of the local system $\pi_{m!}\bar{\mathbf{Q}}_l$ for some m as above. The character sheaves on G are the complexes $IC(G, \mathcal{E})$ for various admissible local systems $\mathcal{E}$ on G_{rs}.

We show how the irreducible characters of $G(\mathbf{F}_q)$ can be recovered from character sheaves on G. If A is a character sheaf on G then its inverse image F^*A under F is again a character sheaf. There are only finitely many A (up to isomorphism) such that F^*A is isomorphic to A. For any such A we choose an isomorphism

Supported in part by the National Science Foundation

Typeset by $\mathcal{AMS}$-TEX

$\phi : F^*A \xrightarrow{\sim} A$ and we form the characteristic function $\chi_{A,\phi} : G(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l$ whose value at g is the alternating sum of traces of ϕ on the stalks at g of the cohomology sheaves of A. Now ϕ is unique up to a non-zero scalar hence $\chi_{A,\phi}$ is unique up to a non-zero scalar. It turns out that

(a) *$\chi_{A,\phi}$ is (up to a non-zero scalar) the character of an irreducible representation of $G(\mathbf{F}_q)$ and $A \mapsto \chi_{A,\phi}$ gives a bijection between the set of (isomorphism classes of) character sheaves on G that are isomorphic to their inverse image under F and the irreducible characters of $G(\mathbf{F}_q)$.*

(This result is essentially contained in [L1,L3].) The main content of this result is that the (rather complicated) values of an irreducible character of $G(\mathbf{F}_q)$ are governed by a geometric principle, namely by the procedure which gives the intersection cohomology extension of a local system.

2. More generally, assume that G is a connected reductive algebraic group over $\mathbf{k}$. The definition of the $IC(G, \mathcal{E})$ given above for GL_n makes sense also in the general case. The complexes on G obtained in this way form the class of *uniform* character sheaves on G. Consider now a fixed $\mathbf{F}_q$-rational structure on G with Frobenius map $F : G \to G$. The analogue of property 1(a) does not hold in general for (G, F). It is still true that the characteristic functions of the uniform character sheaves that are isomorphic to their inverse image under F are linearly independent class functions $G(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l$. However they do not form a basis of the space of class functions. Moreover they are in general not irreducible characters of $G(\mathbf{F}_q)$ (up to a scalar); rather, each of them is a linear combination with known coefficients of a "small" number of irreducible characters of $G(\mathbf{F}_q)$ (where "small" means "bounded independently of q"); this result is essentially contained in [L1,L3].

It turns out that the class of uniform character sheaves can be naturally enlarged to a larger class of complexes on G.

For any parabolic P of G, U_P denotes the unipotent radical of P. For a Borel B in G, the images under $c^B : G \to G/U_B$ of the double cosets BwB form a partition $G/U_B = \cup_w (BwB/U_B)$.

An irreducible intersection cohomology complex $A \in \mathcal{D}(G)$ is said to be a character sheaf on G if it is G-equivariant and if for some/any Borel B in G, $c^B_! A$ has the following property:

$(*)$ *any cohomology sheaf of this complex restricted to any BwB/U_B is a local system with finite monodromy of order invertible in* $\mathbf{k}$.

Then any uniform character sheaf on G is a character sheaf on G. For $G = GL_n$ the converse is also true, but for general G this is not so.

Consider again a fixed $\mathbf{F}_q$-rational structure on G with Frobenius map $F : G \to G$. The following partial analogue of property 1(a) holds (under a mild restriction on the characteristic of $\mathbf{k}$).

(a) *The characteristic functions of the various character sheaves A on G (up to isomorphism) such that $F^*A \xrightarrow{\sim} A$ form a basis of the vector space of class functions $G(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l$.*

3. We now fix a parabolic P of G. For any Borel B of P let $\tilde{c}^B : G/U_P \to G/U_B$ be the obvious map. Now P acts on G/U_P by conjugation.

An irreducible intersection cohomology complex $A \in \mathcal{D}(G/U_P)$ is said to be a parabolic character sheaf if it is P-equivariant and if for some/any Borel B in P, $\tilde{c}^B_! A$ has property 2(∗). When $P = G$, we recover the definition of character sheaves on G.

Consider now a fixed $\mathbf{F}_q$-rational structure on G with Frobenius map $F : G \to G$ such that P is defined over $\mathbf{F}_q$. Then G/U_P has a natural $\mathbf{F}_q$-rational structure with Frobenius map F. The following generalization of 2(a) holds (under a mild restriction on the characteristic of $\mathbf{k}$).

(a) *The characteristic functions of the various parabolic character sheaves A on G/U_P (up to isomorphism) such that $F^*A \xrightarrow{\sim} A$ form a basis of the vector space $\mathcal{V}$ of $P(\mathbf{F}_q)$-invariant functions $G(\mathbf{F}_q)/U_P(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l$.*

The proof is given in [L5]. It relies on a generalization of property 2(a) to not necessarily connected reductive groups which will be contained in the series [L6].

If $h : G(\mathbf{F}_q) \to \bar{\mathbf{Q}}_l$ is the characteristic function of a character sheaf as in 2(a) then by summing h over the fibres of $G(\mathbf{F}_q) \to G(\mathbf{F}_q)/U_P(\mathbf{F}_q)$ we obtain a function $\bar{h} \in \mathcal{V}$. It turns out that each function $\bar{h}$ is a linear combination of a "small" number of elements in the basis of $\mathcal{V}$ described above. (The fact such a basis of $\mathcal{V}$ exists is not apriori obvious.)

The parabolic character sheaves on G/U_P are expected to be a necessary ingredient in establishing the conjectural geometric interpretation of Hecke algebras with unequal parameters given in [L4].

4. In this section G denotes an abelian group with a given family $\mathfrak{F}$ of automorphisms such that

(i) if $F \in \mathfrak{F}$ and $n \in \mathbf{Z}_{>0}$, then $F^n \in \mathfrak{F}$;

(ii) if $F \in \mathfrak{F}, F' \in \mathfrak{F}$ then there exist $n, n' \in \mathbf{Z}_{>0}$ such that $F^n = F'^{n'}$;

(iii) for any $F \in \mathfrak{F}$, the map $G \to G, x \mapsto F(x)x^{-1}$ is surjective with finite kernel.

For $F \in \mathfrak{F}$ and $n \in \mathbf{Z}_{>0}$, the homomorphism

$N_{F^n/F} : G \to G,\ x \mapsto xF(x)\dots F^{n-1}(x),$

restricts to a surjective homomorphism $G^{F^n} \to G^F$. (If $y \in G^F$ we can find $z \in G$ with $y = F^n(z)z^{-1}$, by (i),(iii). We set $x = F(z)z^{-1}$. Then $x \in G^{F^n}$ and $N_{F^n/F}(x) = y$.) Let X be the set of pairs (F, ψ) where $F \in \mathfrak{F}$ and $\psi \in \mathrm{Hom}(G^F, \bar{\mathbf{Q}}_l^*)$. Consider the equivalence relation on X generated by $(F, \psi) \sim (F^n, \psi \circ N_{F^n/F})$. Let G^* be the set of equivalence classes. We define a group structure on G^*. We consider two elements of G^*; we represent them in the form $(F, \psi), (F', \psi')$ where $F = F'$ (using (ii)) and we define their product as the equivalence class of $(F, \psi\psi')$; one checks that this product is independent of the choices. This makes G^* into an abelian group. The unit element is the equivalence class of $(F, 1)$ for any $F \in \mathfrak{F}$. For $F \in \mathfrak{F}$ we define an automorphism $F^* : G^* \to G^*$ by sending an element of G^* represented by (F^n, ψ) with $n \in$

$\mathbf{Z}_{>0}, \psi \in \mathrm{Hom}(G^{F^n}, \bar{\mathbf{Q}}_l^*)$ to $(F^n, \psi \circ F)$ (here $\psi \circ F$ is the composition $G^{F^n} \xrightarrow{F} G^{F^n} \xrightarrow{\psi} \bar{\mathbf{Q}}_l^*$); one checks that this is well defined. For any $F \in \mathfrak{F}$ the map $\mathrm{Hom}(G^F, \bar{\mathbf{Q}}_l^*) \to G^*, \psi \mapsto (F, \psi)$ is

(a) *a group isomorphism of* $\mathrm{Hom}(G^F, \bar{\mathbf{Q}}_l^*)$ *onto the subgroup* $(G^*)^{F^*}$ *of* G^*.
(This follows from the surjectivity of $N_{F^n/F} : G^{F^n} \to G^F$.)

5. Assume now that G is an abelian, connected (affine) algebraic group over $\mathbf{k}$. We define the notion of character sheaf on G.

Let $\mathfrak{F}$ be the set of Frobenius maps $F : G \to G$ for various rational structures on G over a finite subfield of $\mathbf{k}$. (These maps are automorphisms of G as an abstract group.) Then properties 4(i)-4(iii) are satisfied for $(G, \mathfrak{F})$ hence the abelian group G^* is defined as in §4. We will give an interpretation of G^* in terms of local systems on G. Let $F \in \mathfrak{F}$. Let $L : G \to G$ be the Lang map $x \mapsto F(x)x^{-1}$. Consider the local system $E = L_!\bar{\mathbf{Q}}_l$ on G. Its stalk at $y \in G$ is the vector space E_y consisting of all functions $f : L^{-1}(y) \to \bar{\mathbf{Q}}_l$. We have $E_y = \oplus_{\psi \in \mathrm{Hom}(G^F, \bar{\mathbf{Q}}_l^*)} E_y^\psi$ where

$E_y^\psi = \{f \in E_y; f(zx) = \psi(z)f(x) \quad \forall z \in G^F, x \in L^{-1}(y)\}$.

We have a canonical direct sum decomposition $E = \oplus_\psi E^\psi$ where E^ψ is a local system of rank 1 on G whose stalk at $y \in G$ is E_y^ψ (ψ as above). There is a unique isomorphism of local systems $\phi : F^* E^\psi \xrightarrow{\sim} E^\psi$ which induces identity on the stalk at 1. This induces for any $y \in G$ the isomorphism $E_{F(y)}^\psi \to E_y^\psi$ given by $f \mapsto f'$ where $f'(x) = f(F(x))$. If $y \in G^F$, this isomorphism is multiplication by $\psi(y)$. Thus, the characteristic function $\chi_{E^\psi, \phi} : G^F \to \bar{\mathbf{Q}}_l$ is the character ψ.

Let $n \in \mathbf{Z}_{>0}$. Let $L' : G \to G$ be the map $x \mapsto F^n(x)x^{-1}$. Conider the local system $E' = L'_!\bar{\mathbf{Q}}_l$ on G. Its stalk at $y \in G$ is the vector space E'_y consisting of all functions $f' : L'^{-1}(y) \to \bar{\mathbf{Q}}_l$. We define $E_y \to E'_y$ by $f \mapsto f'$ where $f'(x) = f(N_{F^n, F}x)$ (note that $N_{F^n/F}(L'^{-1}(y)) \subset L^{-1}(y)$). This is induced by a morphism of local systems $E \to E'$ which restricts to an isomorphism $E^\psi \xrightarrow{\sim} E'^{\psi'}$ where $\psi' = \psi \circ N_{F^n/F} \in \mathrm{Hom}(G^{F^n}, \bar{\mathbf{Q}}_l^*)$.

¿From the definitions we see that, if $\psi, \psi' \in \mathrm{Hom}(G^F, \bar{\mathbf{Q}}_l^*)$ then for any $y \in G$ we have an isomorphism $E_y^\psi \otimes E_y^{\psi'} \xrightarrow{\sim} E_y^{\psi\psi'}$ given by multiplication of functions on $L^{-1}(y)$. This comes from an isomorphism of local systems $E^\psi \otimes E^{\psi'} \xrightarrow{\sim} E^{\psi\psi'}$.

A *character sheaf* on G is by definition a local system of rank 1 on G of the form E^ψ for some (F, ψ) as above. Let $\mathcal{S}(G)$ be the set of isomorphism classes of character sheaves on G. Then $\mathcal{S}(G)$ is an abelian group under tensor product. The arguments above show that $(F, \psi) \mapsto E^\psi$ defines a (surjective) group homomorphism $G^* \to \mathcal{S}(G)$. This is in fact an isomorphism. (It is enough to show that, if (F, ψ) is as above and $\psi' \in \mathrm{Hom}(G^F, \bar{\mathbf{Q}}_l^*)$ is such that the local systems $E^\psi, E^{\psi'}$ are isomorphic, then $\psi = \psi'$. As we have seen earlier, each of $E^\psi, E^{\psi'}$ has a unique isomorphism ϕ, ϕ' with its inverse image under $F : G \to G$ which induces the identity at the stalk at 1. Then we must have $\chi_{E^\psi, \phi} = \chi_{E^{\psi'}, \phi'}$ hence $\psi = \psi'$. Note that for $F \in \mathfrak{F}$, the map $F^* : G^* \to G^*$ corresponds under the isomorphism

$G^* \xrightarrow{\sim} \mathcal{S}(G)$ to the map $\mathcal{S}(G) \to \mathcal{S}(G)$ given by inverse image under F. Using this and 4(a), we see that, for $F \in \mathfrak{F}$, the map $\mathrm{Hom}(G^F, \bar{\mathbf{Q}}_l^*) \to \mathcal{S}(G), \psi \mapsto E^\psi$ is a group isomorphism of $\mathrm{Hom}(G^F, \bar{\mathbf{Q}}_l^*)$ onto the subgroup of $\mathcal{S}(G)$ consisting of all character sheaves on G that are isomorphis to their inverse image under F. We see that in this case the analogue of 1(a) holds.

¿From the definitions, we see that,

(a) *if* $\mathcal{L}_1 \in \mathcal{S}(G)$ *and* $m : G \times G \to G$ *is the multiplication map then* $m^*\mathcal{L}_1 = \mathcal{L}_1 \otimes \mathcal{L}_1$.

In the case where $G = \mathbf{k}$, our definition of character sheaves on G reduces to that of the Artin-Schreier local systems on $\mathbf{k}$.

6. In this section we assume that G is a unipotent algebraic group over $\mathbf{k}$ of "exponential type" that is, such that the exponential map from Lie G to G is well defined (and an isomorphism of varieties.) In this case we can define character sheaves on G using Kirillov theory. Namely, for each G-orbit in the dual of Lie G we consider the local system $\bar{\mathbf{Q}}_l$ on that orbit extended by 0 on the complement of the orbit. Taking the Fourier-Deligne transform we obtain (up to shift) an irreducible intersection cohomology complex on Lie G (since the orbit is smooth and closed, by Kostant-Rosenlicht). We can view it as an intersection cohomology complex on G via the exponential map. The complexes on G thus obtained are by definition the character sheaves of G. Using Kirillov theory (see [K]) we see that in this case the analogue of 1(a) holds.

Assume, for example, that G is the group of all matrices

$$[a, b, c] = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with entries in $\mathbf{k}$ and that $2^{-1} \in \mathbf{k}$. Consider the following intersection cohomology complexes on G:

(i) the complex which on the centre $\{(0, b, 0); b \in \mathbf{k}\}$ is the local system $\mathcal{E} \in \mathcal{S}(\mathbf{k}), \mathcal{E} \neq \bar{\mathbf{Q}}_l$ wxtended by 0 to the whole of G;

(ii) the local system $f^*\mathcal{E}$ where $f[a, b, c] = (a, c)$ and $\mathcal{E} \in \mathcal{S}(\mathbf{k}^2)$.

The complexes (i),(ii) are the character sheaves of G.

7. In this section we assume that G is a connected unipotent algebraic group over $\mathbf{k}$ (not necessarily of exponential type). We expect that in this case there is again a notion of character sheaf on G such that over a finite field, the characteristic functions of character sheaves form a basis of the space of class functions and each characteristic function of a character sheaf is a linear combination of a "small" number of irreducible characters. Thus here the situation should be similar to that for a general connected reductive group rather than that for GL_n. We illustrate this in one example. Assume that $\mathbf{k}$ has characteristic 2. Let G be the group consisting of all matrices of the form

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & b+ad \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with entries in $\mathbf{k}$; we also write $[a,b,c,d]$ instead of the matrix above. (This group can be regarded as the unipotent radical of a Borel in $Sp_4(\mathbf{k})$.)

Let $\mathcal{E}_0 \in \mathcal{S}(\mathbf{k})$ be the local system on $\mathbf{k}$ associated in §5 to $\mathbf{F}_q$ and to the homomorphism $\psi_0 : \mathbf{F}_q \to \bar{\mathbf{Q}}_l^*$ (composition of the trace $\mathbf{F}_q \to \mathbf{F}_2$ and the unique injective homomorphism $\mathbf{F}_2 \to \bar{\mathbf{Q}}_l^*$).

Consider the following intersection cohomology complexes on G:

(i) the complex which on the centre $\{[0,b,c,0]; (b,c) \in \mathbf{k}^2\}$ is the local system $\mathcal{E} \in \mathcal{S}(\mathbf{k}^2), \mathcal{E} \neq \bar{\mathbf{Q}}_l$ (see §5) extended by 0 to the whole of G;

(ii) the complex which on $\{[a_0,b,c,0]; (b,c) \in \mathbf{k}^2\}$ (with $a_0 \in \mathbf{k}^*$ fixed) is the local system $pr_c^*\mathcal{E}$ where $\mathcal{E} \in \mathcal{S}(\mathbf{k}), \mathcal{E} \neq \bar{\mathbf{Q}}_l$ (see §5) extended by 0 to the whole of G;

(iii) the complex which on $\{[0,b,c,d_0]; (b,c) \in \mathbf{k}^2\}$ (with $d_0 \in \mathbf{k}^*$ fixed) is the local system $f^*\mathcal{E}_0$ where $f[0,b,c,d_0] = \alpha b + \alpha^2 d_0 c$ (with $\alpha \in \mathbf{k}^*$ fixed) extended by 0 to the whole of G;

(iv) the complex which on $\{[a_0,b,c,d_0]; (b,c) \in \mathbf{k}^2\}$ (with $a_0, d_0 \in \mathbf{k}^*$ fixed) is the local system $f^*\mathcal{E}_0$ where $f[a_0,b,c,d_0] = a_0^{-2} d_0^{-1} c$ extended by 0 to the whole of G;

(v) the local system $f^*\mathcal{E}$ on G where $f[a,b,c,d] = (a,d) \in \mathbf{k}^2$ and $\mathcal{E} \in \mathcal{S}(\mathbf{k}^2)$.

By definition, the character sheaves on G are the complexes in (i)-(v) above. Note that there are infinitely many subvarieties of G which appear as supports of character sheaves (this in contrast with the case of reductive groups). There is a symmetry that exchanges the character sheaves of type (ii) with those of type (iii). Namely, define $\xi : G \to G$ by

$$[a,b,c,d] \mapsto [d, c+ab+a^2d, b^2+dc+abd, a^2].$$

Then ξ is a homomorphism whose square is $[a,b,c,d] \mapsto [a^2,b^2,c^2,d^2]$; moreover, ξ^* interchanges the sets (ii) and (iii) and it leaves stable each of the sets (i), (iv) and (v).

Now G has an obvious $\mathbf{F}_q$-structure with Frobenius map $F : G \to G$. We describe the irreducible characters of $G(\mathbf{F}_q)$.

(I) We have q^2 one dimensional characters $U \to \bar{\mathbf{Q}}_l^*$ of the form $[a,b,c,d] \mapsto \psi_0(xa+yd)$ (one for each $x, y \in \mathbf{F}_q$).

(II) We have $q-1$ irreducible characters of degree q of the form $[0,b,c,0] \mapsto q\psi_0(xb)$ (all other elements are mapped to 0), one for each $x \in \mathbf{F}_q - \{0\}$.

(III) We have $q-1$ irreducible characters of degree q of the form $[0,b,c,0] \mapsto q\psi_0(xc)$ (all other elements are mapped to 0), one for each $x \in \mathbf{F}_q - \{0\}$.

(IV) We have $4(q-1)^2$ irreducible characters of degree $q/2$, one for each quadruple $(a_0, d_0, \epsilon_1, \epsilon_2)$ where
$a_0 \in \mathbf{F}_q^*, d_0 \in \mathbf{F}_q^*, \epsilon_1 \in \mathrm{Hom}(\{0, a_0\}, \pm 1), \epsilon_2 \in \mathrm{Hom}(\{0, d_0\}, \pm 1)$,
namely
$[a, b, c, d] \mapsto (q/2)\epsilon_1(a)\epsilon_2(d)\psi_0(a_0^{-2}d_0^{-1}(ba + ba_0 + c))$,
if $a \in \{0, a_0\}, d \in \{0, d_0\}$; all other elements are sent to 0.
A character of type (II) is obtained by inducing from the subgroup $\{[a, b, c, d] \in G(\mathbf{F}_q); d = 0\}$ the one dimensional character $[a, b, c, 0] \mapsto \psi_0(xb)$ where $x \in \mathbf{F}_q - \{0\}$. A character of type (III) is obtained by inducing from the commutative subgroup $\{[a, b, c, d] \in G(\mathbf{F}_q); a = 0\}$ the one dimensional character $[0, b, c, d] \mapsto \psi_0(xc)$ where $x \in \mathbf{F}_q - \{0\}$. A character of type (IV) is obtained by inducing from the subgroup $\{(a, b, c, d) \in G(\mathbf{F}_q); a \in \{0, a_0\}\}$ (where $a_0 \in \mathbf{F}_q - \{0\}$ is fixed) the one dimensional character $[a, b, c, d] \mapsto \epsilon_1(a)\psi_0(fd + a_0^{-2}d_0^{-1}(ba + ba_0 + c))$ where $f \in \mathbf{F}_q$ is chosen so that $\psi_0(fd_0) = \epsilon_2(d_0)$ (the induced character does not depend on the choice of f).

Consider the matrix expressing the characteristic functions of character sheaves A such that $F^*A \cong A$ (suitably normalized) in terms of irreducible characters of $G(\mathbf{F}_q)$. This matrix is square and a direct sum of diagonal blocks of size 1×1 (with entry 1) or 4×4 with entries $\pm 1/2$, representing the Fourier transform over a two dimensional symplectic $\mathbf{F}_2$-vector space. There are $(q-1)^2$ blocks of size 4×4 involving the irreducible characters of type IV.

We see that, in our case, the character sheaves have the desired properties. We also note that in our case, $G(\mathbf{F}_q)$ has some irreducible character whose degree is not a power of q (but $q/2$) in contrast with what happens in the situation in §6.

8. Let ϵ be an indeterminate. For $r \geq 2$ let $\mathcal{A}_r = \mathbf{k}[\epsilon]/(\epsilon^r)$. Let $G = GL_n(\mathcal{A}_r)$. Let B (resp.T) be the group of upper triangular (resp. diagonal) matrices in G. Then G is in a natural way a connected affine algebraic group over $\mathbf{k}$ of dimension n^2r and B, T are closed subgroups of G. On G we have a natural $\mathbf{F}_q$-structure with Frobenius map $F : G \to G$, $(g_{ij}) \mapsto (g_{ij}^{(q)})$ where for $a_0, a_1, \dots, a_{r-1}$ in $\mathbf{k}$ we set $(a_0 + a_1\epsilon + \dots + a_{r-1}\epsilon^{r-1})^{(q)} = a_0^q + a_1^q\epsilon + \dots + a_{r-1}^q\epsilon^{r-1}$. The fixed point set of $F : G \to G$ is $GL_n(\mathbf{F}_q[\epsilon]/(\epsilon^r))$. For $i \neq j$ in $[1, n]$, we consider the homomorphism $f_{ij} : \mathbf{k} \to T$ which takes $x \in \mathbf{k}$ to the diagonal matrix with ii-entry equal to $1 + \epsilon^{r-1}x$, jj-entry equal to $1 - \epsilon^{r-1}x$ and other diagonal entries equal to 1. Since T is connected and commutative, the group $\mathcal{S}(T)$ is defined (see §5). Let $\mathcal{L} \in \mathcal{S}(T)$. We will assume that $\mathcal{L}$ is *regular* in the following sense: for any $i \neq j$ in $[1, n]$, $f_{ij}^*\mathcal{L}$ is not isomorphic to $\bar{\mathbf{Q}}_l$.

Let $\pi : B \to T$ be the obvious homomorphism. Consider the diagram

$$G \xleftarrow{a} Y \xrightarrow{b} T$$

where

$$Y = \{(g, xB) \in G \times G/B; x^{-1}gx \in B\}, a(g, xB) = g, b(g, xB) = \pi(x^{-1}gx).$$

Then $b^*\mathcal{L}$ is a local system on Y and we may consider the complex $a_!b^*\mathcal{L}$ on G.

As in §5, we can find an integer $m_0 > 0$ such that, for any $m \in \mathcal{M} = \{m_0, 2m_0, 3m_0, \dots\}$, $\mathcal{L}$ is associated to $(\mathbf{F}_{q^m}, \psi_m)$ where $\psi_m \in \mathrm{Hom}(T^{F^m}, \bar{\mathbf{Q}}_l^*)$. We can regard ψ_m as a character $B(\mathbf{F}_{q^m}) \to \bar{\mathbf{Q}}_l^*$ via $\pi : B \to T$; inducing this from $B(\mathbf{F}_{q^m})$ to $G(\mathbf{F}_{q^m})$ we obtain a representation of $G(\mathbf{F}_{q^m})$ whose character is denoted by c_m. It is easy to see (using the regularity of $\mathcal{L}$) that this character is irreducible.

For $m \in \mathcal{M}$, there is a unique isomorphism $(F^m)^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ of local systems on T which induces the identity on the stalk of $\mathcal{L}$ at 1. This induces an isomorphism $(F^m)^*(b^*\mathcal{L}) \xrightarrow{\sim} b^*\mathcal{L}$ (where $F : Y \to Y$ is $(g, xB) \mapsto (F(g), F(x)B)$) and an isomorphism $(F^m)^*(a_!b^*\mathcal{L}) \xrightarrow{\sim} a_!b^*\mathcal{L}$ in $\mathcal{D}(G)$. Let $\chi_m : G^{F^m} \to \bar{\mathbf{Q}}_l$ be the characteristic function of $a_!b^*\mathcal{L}$ with respect to this isomorphism. From the definitions we see that $\chi_m = c_m$. This shows that $a_!b^*\mathcal{L}$ behaves like a character sheaf except for the fact that it is not clear that it is an intersection cohomology complex.

We conjecture that:

(a) *if $\mathcal{L}$ is regular then $a_!b^*\mathcal{L}$ is an intersection cohomology complex on G.*

(The conjecture also makes sense and is expected to be true when GL_n is replaced by any reductive group, and G by the corresponding group over $\mathcal{A}_r$.) Thus one can expect that there is a theory of character sheaves for G, as far as generic principal series representations and their twisted forms is concerned. But one cannot expect a complete theory of character sheaves in this case (see §13).

In §9-§12 we prove the conjecture in the special case where $G = GL_2(\mathbf{k})$ and $r = 2$.

9. Let $\mathcal{A} = \mathcal{A}_2 = \mathbf{k}[\epsilon]/(\epsilon^2)$. Let V be a free $\mathcal{A}$-module of rank 2. Let G be the group of automorphisms of the $\mathcal{A}$-module V. This is the group of all automorphisms of the 4-dimensional $\mathbf{k}$-vector space V that commute with the map $\epsilon : V \to V$ given by the $\mathcal{A}$-module structure. Hence G is an algebraic group of dimension 8 over $\mathbf{k}$. Let ${}^0\tilde{G}$ be the set of all pairs (g, V_2) where $g \in G$ and V_2 is a free $\mathcal{A}$-submodule of V of rank 1 such that $gV_2 = V_2$. For $k = 1, 2$, let X_k be the set of all $\mathcal{A}$-submodules of V that have dimension k as a $\mathbf{k}$-vector space. Let $\tilde{G}$ be the set of all triples (g, V_1, V_2) where $g \in G$, $V_1 \in X_1, V_2 \in X_2$, $V_1 \subset V_2$, $gV_1 = V_1, gV_2 = V_2$ and the scalars by which g acts on V_1 and V_2/V_1 coincide. We can regard ${}^0\tilde{G}$ as a subset of $\tilde{G}$ by $(g, V_2) \mapsto (g, \epsilon V_2, V_2)$. Note that $\tilde{G}$ is naturally an algebraic variety over $\mathbf{k}$ and ${}^0\tilde{G}$ is an open subset of $\tilde{G}$.

The group of units $\mathcal{A}'$ of $\mathcal{A}$ is an algebraic group isomorphic to $\mathbf{k}^* \times \mathbf{k}$. Hence $\mathcal{S}(\mathcal{A}')$ is defined. Let $\mathcal{L}_1 \in \mathcal{S}(\mathcal{S}'), \mathcal{L}_2 \in \mathcal{S}(\mathcal{S}')$. Let $\mathcal{L} = \mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{S}(\mathcal{A}' \times \mathcal{A}')$, $\mathcal{E} = \mathcal{L}_2 \otimes \mathcal{L}_1^* \in \mathcal{S}(\mathcal{A}')$. Define $f : {}^0\tilde{G} \to \mathcal{A}' \times \mathcal{A}'$ by $f(g, V_2) = (\alpha_1, \alpha_2)$ where $\alpha_1 \in \mathcal{A}'$ is given by $gv = \alpha_1 v$ for $v \in V_2$ and $\alpha_2 \in \mathcal{A}'$ is given by $gv' = \alpha_2 v'$ for $v' \in V/V_2$. Let $\tilde{\mathcal{L}} = f^*(\mathcal{L}_1 \boxtimes \mathcal{L}_2)$, a local system on ${}^0\tilde{G}$. Define $f_i : {}^0\tilde{G} \to \mathcal{A}'$ $(i = 1, 2)$ by $f_1(g, V_2) = \alpha_1\alpha_2$, $f_2(g, V_2) = \alpha_1$ where α_1, α_2 are as above. Then $\tilde{\mathcal{L}} = f_1^*\mathcal{L}_1 \otimes f_2^*\mathcal{L}$. (We use 5(a).)

We shall assume that $\mathcal{L}$ is *regular* in the following sense: the restriction of $\mathcal{E}$ to the subgroup $\mathcal{T} = \{1 + \epsilon c; c \in \mathbf{k}\}$ of $\mathcal{A}'$ is not isomorphic to $\bar{\mathbf{Q}}_l$.

Lemma 10. *(a) $\tilde{G}$ is an irreducible, smooth variety and $\tilde{G} - {}^0\tilde{G}$ is a smooth irreducible hypersurface in $\tilde{G}$.*

(b) We have $IC(\tilde{G}, \tilde{\mathcal{L}})|_{\tilde{G}-{}^0\tilde{G}} = 0$.

Note that $f_1 : {}^0\tilde{G} \to \mathcal{A}'$ extends to the whole of $\tilde{G}$ by $f_1(g, V_1, V_2) = \det_{\mathcal{A}}(g : V \to V)$. Hence $f_1^*\mathcal{L}_1$ extends to a local system on $\tilde{G}$ and we have $IC(\tilde{G}, \tilde{\mathcal{L}}) = f_1^*\mathcal{L}_1 \otimes IC(\tilde{G}, f_2^*\mathcal{E})$. Hence to prove (b) it is enough to show that $IC(\tilde{G}, f_2^*\mathcal{E})$ is zero on $\tilde{G} - {}^0\tilde{G}$.

Let Z (resp. H) be the fibre of the second projection $\tilde{G} \to X_1$ (resp. $\tilde{G} - {}^0\tilde{G} \to X_1$) at $V_1 \in X_1$. Since G acts transitively on X_1 it is enough to show that Z is smooth, irreducible, H is a smooth, irreducible hypersurface in Z and $IC(Z, f_2^*\mathcal{E})$ is zero on H (the restriction of f_2 to Z is denoted again by f_2).

Let e_1, e_2 be a basis of V such that $V_1 = \mathbf{k}\epsilon e_1$. The subspaces $V_2 \in X_2$ such that $V_1 \subset V_2$ are exactly the subspaces $V_2^{z',z''} = \mathbf{k}\epsilon e_1 + \mathbf{k}(z'e_1 + z''\epsilon e_2)$ where $(z', z'') \in \mathbf{k}^2 - \{0\}$. An element $g \in G$ is of the form

$$ge_1 = a_0e_1 + b_0e_2 + a_1\epsilon e_1 + b_1\epsilon e_2,$$
$$ge_2 = c_0e_1 + d_0e_2 + c_1\epsilon e_1 + d_1\epsilon e_2$$

where $a_i, b_i, c_i, d_i \in \mathbf{k}$ satisfy $a_0d_0 - b_0c_0 \neq 0$.

The condition that $g\epsilon e_1 \in \mathbf{k}\epsilon e_1$ is $b_0 = 0$. The condition that $gV_2^{z',z''} = V_2^{z',z''}$ is that $z'b_1 + z''d_0 = a_0z''$ if $z' \neq 0$ (no condition if $z' = 0$). The condition that the scalars by which g acts on V_1 and $V_2^{z',z''}/V_1$ coincide is $a_0 = d_0$ if $z' = 0$ (no condition if $z' \neq 0$).

We see that we may identify Z with

$$\{(a_0, c_0, d_0, a_1, b_1, c_1, d_1; z', z'') \in \mathbf{k}^7 \times (\mathbf{k}^2 - \{0\})/\mathbf{k}^*; \\ a_0 \neq 0, d_0 \neq 0, z'b_1 = z''(a_0 - d_0)\}$$

and H with the subset defined by $z' = 0$. In this description it is clear that Z is irreducible, smooth and H is a smooth, irreducible hypersurface in Z. The function f_2 takes a point with $z' \neq 0$ to $a_0 + \epsilon(a_1 + z''z'^{-1}c_0)$. To prove the statement on intersection cohomology we may replace Z by the open subset $z'' \neq 0$ containing H. Thus we may replace Z by

$$Z_1 = \{(a_0, c_0, d_0, a_1, b_1, c_1, d_1; z) \in \mathbf{k}^7 \times \mathbf{k}; a_0 \neq 0, d_0 \neq 0, zb_1 = a_0 - d_0\}$$

and H by the subset defined by $z = 0$. The function f_2 is defined on $Z_1 - H$ by

$$a_0 + \epsilon(a_1 + z^{-1}c_0) = (a_0 + \epsilon a_1)(1 + \epsilon z^{-1}c_0a_0^{-1}).$$

Thus $f_2 = f_3f_4$ where f_3 (resp. f_4) is defined on $Z_1 - H$ by $a_0 + \epsilon a_1$ (resp. $1 + \epsilon z^{-1}c_0a_0^{-1}$). Hence $f_2^*\mathcal{E} = f_3^*\mathcal{E} \otimes f_4^*\mathcal{E}$. Now f_3 extends to Z_1 hence $f_3^*\mathcal{E}$ extends to a local system on Z_1. We have $IC(Z_1, f_3^*\mathcal{E} \otimes f_4^*\mathcal{E}) = f_3^*\mathcal{E} \otimes IC(Z_1, f_4^*\mathcal{E})$. It is enough to show that $IC(Z_1, f_4^*\mathcal{E})$ is zero on H. We make the change of variable $c = c_0 a_0^{-1}$. Then Z_1 becomes

$$Z_1 = \{(a_0, c, a_1, b_1, c_1, d_1; z) \in \mathbf{k}^7 \times \mathbf{k}; a_0 \neq 0, a_0 - zb_1 \neq 0\},$$

H is the subset defined by $z = 0$ and $f_4 : Z_1 - H \to \mathcal{A}'$ is given by $1 + \epsilon z^{-1} c$. Let $\tilde{Z}_1 = \{(a_0, c, a_1, b_1, c_1, d_1; z) \in \mathbf{k}^7 \times \mathbf{k}\}$ and let H_1 be the subset of $\tilde{Z}_1$ defined by $z = 0$. Then Z_1 is open in $\tilde{Z}_1$ and f_4 is well defined on $\tilde{Z}_1 - H_1$ by $1 + \epsilon z^{-1} c$. Hence $f_4^*\mathcal{E}$ is well defined on $\tilde{Z}_1 - H_1$. It is enough to show that $IC(\tilde{Z}_1, f_4^*\mathcal{E})$ is zero on H_1. Let $H' = \{(c, z) \in \mathbf{k}^2; z = 0\}$ and define $f' : \mathbf{k}^2 - H' \to \mathcal{A}'$ by $f'(c, z) = 1 + \epsilon z^{-1} c$. It is enough to show that $IC(\mathbf{k}^2, f'^*\mathcal{E})$ is zero on H'. Let P be the projective line associate to $\mathbf{k}^2$. Then H' defines a point $x_0 \in P$. Since f' is constant on lines, it defines a map $h : P - \{x_0\} \to \mathcal{A}'$. Since P is 1-dimensional we have $IC(P, h^*\mathcal{E}) = \mathcal{F}$ where $\mathcal{F}$ is a constructible sheaf on P whose restriction to $P - \{x_0\}$ is $h^*\mathcal{E}$. It is enough to show that

(c) the stalk of $\mathcal{F}$ at x_0 is 0;

(d) $H^i(P, \mathcal{F}) = 0$ for $i = 0, 1$.

(Indeed, (c) implies that $IC(\mathbf{k}^2, f'^*\mathcal{E})$ is zero at $(c, 0)$ with $c \neq 0$ and (d) implies that $IC(\mathbf{k}^2, f'^*\mathcal{E})$ is zero at $(0, 0)$.)

Consider the standard $\mathbf{F}_q$-rational structures an $\mathbf{k}^2, X, \mathcal{A}'$ and let F be the corresponding Frobenius map. We may assume that $\mathcal{E}$ is associated as in §5 to $(\mathbf{F}_q, \psi)$ where $\psi \in \mathrm{Hom}(\mathcal{A}'^F, \bar{\mathbf{Q}}_l^*)$. For any $m \in \mathbf{Z}_{>0}$ there is a unique isomorphism $\phi_m : (F^m)^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ which induces the identity on the stalk of $\mathcal{E}$ at 1. The characteristic function of $\mathcal{E}$ with respect to this isomorphism is $a' \mapsto \psi(N_{F^m/F}(a'))$, $a' \in \mathcal{A}'^{F^m}$. Since, by assumption, $\mathcal{E}|_{\mathcal{T}}$ is not isomorphic to $\bar{\mathbf{Q}}_l$, $\psi|_{\mathcal{T}^F}$ is not the trivial character. Hence $\psi \circ N_{F^m/F} : \mathcal{A}'^{F^m} \to \bar{\mathbf{Q}}_l^*$ is non-trivial on $\mathcal{T}^{F^m}$. Now ϕ_m induces an isomorphism $\phi'_m : (F^m)^* h^*\mathcal{E} \xrightarrow{\sim} h^*\mathcal{E}$. We show that

(e) $\sum_{x \in P^{F^m} - \{x_0\}} \mathrm{tr}(\phi'_m, (h^*\mathcal{E})_x) = 0$.

An equivalent statement is:

$\sum_{(c,z) \in \mathbf{F}_{q^m} \times \mathbf{F}_{q^m}^*} (\psi \circ N_{F^m/F})(1 + \epsilon z^{-1} c) = 0$,

which follows from the fact that $\psi \circ N_{F^m/F} : \mathcal{A}'^{F^m} \to \bar{\mathbf{Q}}_l^*$ is non-trivial on $\mathcal{T}^{F^m}$. Introducing (e) in the trace formula for Frobenius, we see that

(f) $\sum_{i=0}^{2} (-1)^i \mathrm{tr}(\phi'_m, H^i(P, \mathcal{F})) = \mathrm{tr}(\phi'_m, \mathcal{F}_{x_0})$

where $\mathcal{F}_{x_0}$ is the talk of $\mathcal{F}$ at x_0 and ϕ'_m is in fact equal to ${\phi'_1}^m$ (for $m = 1, 2, 3, \dots$). By Deligne's purity theorem, $H^i(P, \mathcal{F})$ together with ϕ'_1 is pure of weight i; by Gabber's theorem [BBD], $\mathcal{F}_{x_0}$ together with ϕ'_1 is mixed of weight ≤ 0. Hence from (f) we deduce that $H^1(P, \mathcal{F}) = 0, H^2(P, \mathcal{F}) = 0$ and $\dim H^0(P, \mathcal{F}) = \dim \mathcal{F}_{x_0}$. By the hard Lefschetz theorem [BBD] we have $\dim H^0(P, \mathcal{F}) = \dim H^2(P, \mathcal{F})$. It follows that $H^0(P, \mathcal{F}) = 0$ hence $\mathcal{F}_{x_0} = 0$. This proves (c),(d). The lemma is proved.

Lemma 11. *Define $\rho : {}^0\tilde{G} \to G$ by $(g, V_2) \mapsto g$. Let $K = \rho_! \tilde{\mathcal{L}}$. Let G_0 be the open dense subset of G consisting of all $g \in G$ such that $g : \epsilon V \to \epsilon V$ is regular,* *semisimple. Let* $\rho_0 : \rho^{-1}(G_0) \to G_0$ *be the restriction of* ρ*. Then* $\rho_{0!}\tilde{\mathcal{L}}$ *is a local system on* G_0*. We have* $\dim \mathrm{supp}\mathcal{H}^i K < \dim G - i$ *for any* $i > 0$*.*

The first assertion of the lemma follows from the fact that ρ_0 is a double covering. To prove the second assertion it is enough to show that, for $i > 0$, the set G_i consisting of the points $g \in G$ such that $\dim \rho^{-1}(g) = i$ and $\oplus_j H_c^j(\rho^{-1}(g), \tilde{\mathcal{L}}) \neq 0$ has codimension $> 2i$ in G.

Consider the fibre $\rho^{-1}(g)$ for $g \in G$. We may assume that, with respect to a suitable $\mathcal{A}$-basis of V, g can be represented as an upper triangular matrix $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with a, c in $\mathcal{A}'$ and $b \in \mathcal{A}$. (Otherwise, $\rho^{-1}(g)$ is empty.) There are five cases:

Case 1. $a - d \in \mathcal{A}'$. Then $\rho^{-1}(g)$ consists of two points.

Case 2. $a - d \in \epsilon\mathcal{A}, b \in \mathcal{A}'$. Then $\rho^{-1}(g)$ is an affine line.

Case 3. $a - d \in \epsilon\mathcal{A} - \{0\}, b \in \epsilon\mathcal{A}$. Then $\rho^{-1}(g)$ is a disjoint union of two affine lines.

Case 4. $a = d, b \in \epsilon\mathcal{A} - \{0\}$. Then $\rho^{-1}(g)$ is an affine line.

Case 5. $a = d, b = 0$. Then $\rho^{-1}(g)$ is an affine line bundle over a projective line.

In case 2, we may identify $\rho^{-1}(g), \tilde{\mathcal{L}}|_{\rho^{-1}(g)}$ with $P - \{x_0\}, \mathcal{F}|_{P-\{x_0\}}$ in the proof of Lemma 10. Then the argument in that proof shows that $H_c^j(\rho^{-1}(g), \tilde{\mathcal{L}}) = 0$ for all j. We see that G_1 consists of all g as in case 3 and 4, hence G_1 has codimension 3 in G. We see that G_2 consists of all g as in case 5, hence G_2 has codimension 6 in G. The lemma is proved. Note that without the assumption that $\mathcal{L}$ is regular, the last assertion of the lemma would not hold (there would be a violation coming from g in case 2.)

12. We show:

(a) $$\rho_!\tilde{\mathcal{L}} = IC(G, \rho_{0!}\tilde{\mathcal{L}}).$$

Define $\tilde{\rho} : \tilde{G} \to G$ by $\tilde{\rho}(g, V_1, V_2) = g$. Clearly, $\tilde{\rho}$ is proper. Let $j : {}^0\tilde{G} \to G$ be the inclusion. We have $\rho = \tilde{\rho} \circ j$ hence $\rho_!\tilde{\mathcal{L}} = \tilde{\rho}_!(j_!\tilde{\mathcal{L}})$. By Lemma 10, we have $j_!\tilde{\mathcal{L}} = IC(\tilde{G}, \tilde{\mathcal{L}})$ hence $\rho_!\tilde{\mathcal{L}} = \tilde{\rho}_!IC(\tilde{G}, \tilde{\mathcal{L}})$. Since $\tilde{\rho}$ is proper, $\tilde{\rho}_!$ commutes with the Verdier duality $\mathfrak{D}$. Hence $\mathfrak{D}(\rho_!\tilde{\mathcal{L}}) = \tilde{\rho}_!\mathfrak{D}IC(\tilde{G}, \tilde{\mathcal{L}})$. Hence $\mathfrak{D}(\rho_!\tilde{\mathcal{L}})$ equals $\tilde{\rho}_!IC(\tilde{G}, \tilde{\mathcal{L}}^*)$ up to a shift. Now the same argument that shows $j_!\tilde{\mathcal{L}} = IC(\tilde{G}, \tilde{\mathcal{L}})$ shows also $j_!\tilde{\mathcal{L}}^* = IC(\tilde{G}, \tilde{\mathcal{L}}^*)$. Hence, up to shift, $\mathfrak{D}(\rho_!\tilde{\mathcal{L}})$ equals $\tilde{\rho}_!j_!\tilde{\mathcal{L}}^* = \rho_!\tilde{\mathcal{L}}^*$. Now the argument in Lemma 12 can also be applied to $\tilde{\mathcal{L}}^*$ instead of $\tilde{\mathcal{L}}$ and yields $\dim \mathrm{supp}\mathcal{H}^i\rho_!\tilde{\mathcal{L}}^* < \dim G - i$ for any $i > 0$. Thus, $\rho_!\tilde{\mathcal{L}}$ satisfies the defining properties of $IC(G, \rho_{0!}\tilde{\mathcal{L}})$ hence it is equal to it. This proves (a).

We see that conjecture 8(a) holds for $n = 2, r = 2$.

13. If G is a connected affine algebraic group over $\mathbf{k}$ which is neither reductive nor nilpotent, one cannot expect to have a complete theory character sheaves for G. Assume for example that G is the group of all matrices

$$[a, b] = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with entries in $\mathbf{k}$. The group $G(\mathbf{F}_q)$ (for the obvious $\mathbf{F}_q$-rational structure) has $(q-1)$ one dimensional representations and one $(q-1)$-dimensional irreducible representation. The character of a one dimensional representation can be realized in terms of an intersection cohomology complex (a local system on G), but that of the $(q-1)$ dimensional irreducible representation appears as a difference of two intersection cohomology complexes, one given by the local system $\bar{\mathbf{Q}}_l$ on the unipotent radical of G and one supported by the unit element of G. A similar phenomenon occurs for G as in §9 and for a (q^2-1)-dimensional irreducible representation of $G(\mathbf{F}_q)$.

References

[BBD] A.A.Beilinson, J.Bernstein and P.Deligne, *Faisceaux pervers*, Astérisque **100** (1982).

[G] J.A.Green, *The characters of the finite general linear groups*, Trans.Amer.Math.Soc. **80** (1955), 402-447.

[K] D.Kazhdan, *Proof of Springer's hypothesis*, Israel J.Math. **28** (1977), 272-286.

[L1] G.Lusztig, *Characters of reductive groups over a finite field*, Ann.Math.Studies 107, Princeton U.Press, 1984.

[L2] G.Lusztig, *Character sheaves, I-V*, Adv.in Math. **56,57,59,61** (1985,1986).

[L3] G.Lusztig, *Green functions and character sheaves*, Ann.Math. **131** (1990), 355-408.

[L4] G.Lusztig, *Hecke algebras with unequal parameters*, CRM Monographs Series, vol. 18, Amer.Math.Soc., 2003.

[L5] G.Lusztig, *Parabolic character sheaves, I*, Moscow Math.J. (2003) (to appear).

[L6] G.Lusztig, *Character sheaves on disconnected groups, I-III* (2003), preprints.

Department of Mathematics, M.I.T., Cambridge, MA 02139