V in the form of the postulate which in the literature on the history of mathematics is sometimes termed the postulate of Eudoxus (with regard to two unequal magnitudes a and $b (a < b)$ it is assumed that there exists a number n having the property that $n \cdot a > b$). But on this postulate are based both the whole theory of proportions of Euclid V, which Archimedes makes use of all through his work, and Euclid's lemma (X, 1) about the continued dichotomy of a magnitude, also regularly used by Archimedes. Why then should he have formulated it once more in a separate assumption for a magnitude appearing as the difference between two other magnitudes?

The actual meaning, however, becomes plain when it is considered that in Greek mathematics there existed, side by side with the strict and official method of the indirect passage to a limit, also the less strict, but heuristically more fertile method of indivisibles, and that Archimedes himself diligently used it as a method of investigation[1]). In this method a solid is regarded as the sum of plane sections, a surface as the sum of lines, while it also includes the view of a curve as being generated by juxtaposition of points; this might easily suggest the idea that thus the difference between two solids might be a surface, that between two surfaces a length. We may ignore the question whether this idea was ever applied in practice. But it is in any case understandable that Archimedes, now that he is about to prove his results strictly, with the aid of a method the essential foundation of which is precisely that the difference between two magnitudes of the same kind, however small it may be, satisfies the axiom of Eudoxus with regard to any magnitude of this kind, deems it necessary to banish all unstrict conceptions to which the method of indivisibles might give rise on this point.

From the above it follows that it is desirable to discriminate explicitly between the axiom of Eudoxus and the 5th postulate of the treatise *On the Sphere and Cylinder* of Archimedes. The contents of the latter may be briefly summed up as follows: if two magnitudes satisfy the axiom of Eudoxus in respect of each other, their difference also satisfies this assumption in respect of any magnitude of the same kind homogeneous with both[2]).

[1]) Vide Chapter X.

[2]) J. Hjelmslev, in his treatise *Über Archimedes' Grössenlehre* (Det kgl.

To express it in modern terms, he excludes the existence of actual infinitesimals; the magnitudes he is going to discuss are to form Eudoxian systems.

At the end of the Lambanomena it is mentioned, by way of conclusion from the second, that the perimeter of a polygon inscribed in a circle is less than the circumference of the circle.

4. *Introductory Propositions* (1–6).

In the first Book of *On the Sphere and Cylinder* the ratio form of the compression method (III; 8, 21) is to be repeatedly applied. The group of the propositions 2–6 serves to prepare the way. It is preceded by the proposition 1, in which it is derived from the second assumption that the perimeter of a polygon circumscribed about a circle is greater than the circumference of the circle.

In the following propositions

C denotes a circle, C_n a regular polygon of n sides circumscribed about this circle, I_n a regular polygon of n sides inscribed in this circle. All three symbols at the same time denote the area of the figures they represent. The sides of the polygons are called successively Z_n and z_n.

Danske Videnskabernes Selskab. Matem.-Fysiske Meddelelser XXV, No 15, København 1950, pp. 4, 5), distinguishes the two axioms in question as the axiom of Eudoxus and the lemma of Archimedes. According to him, the object of the lemma is to establish that, when two magnitudes satisfy the axiom of Eudoxus in respect of each other, their difference also satisfies it in respect of all magnitudes of the same kind with a and b. This view is in agreement with the one defended above; it differs from it only in the motivation: the formulation of the new axiom is considered necessary not for the sake of excluding the method of indivisibles, but to give sense to the difference of two homogeneous magnitudes a and b; *e.g.* in the case where a is a circular arc and b a line segment, or a part of the surface of a sphere and b part of a plane. In the theory of proportions of Euclid this axiom, according to the author's view, was not necessary, because $a - b$ always exists as a magnitude of the same kind with a and b. We are not convinced by this argument. Eudoxus (in Euclid V) merely requires of his magnitudes that they shall satisfy his axiom, and does not say at all what magnitudes they are. It cannot be understood why with him a could not be a circular arc and b a line segment.

The axiom of Archimedes is not therefore required because the scope of the geometrical magnitudes under consideration is widened, but it serves to fill up a gap in the theory of proportions of Euclid V (Euclid, for example, tacitly assumes in V, 8 what the axiom of Archimedes explicitly postulates). In fact, through this gap the indivisibles might slip into geometry again.

V. Ἔτι δὲ τῶν ἀνίσων γραμμῶν καὶ τῶν ἀνίσων ἐπιφανειῶν καὶ τῶν ἀνίσων στερεῶν τὸ μεῖζον τοῦ ἐλάσσονος ὑπερέχειν τοιούτῳ, ὃ συντιθέμενον αὐτὸ ἑαυτῷ δυνατόν ἐστιν ὑπερέχειν παντὸς τοῦ προτεθέντος τῶν πρὸς ἄλληλα λεγομένων.

V. *And also that of unequal lines, unequal surfaces, and unequal solids the greater exceeds the lesser by an amount such that, when added to itself, it may exceed any assigned magnitude of the type of magnitudes compared with one another.*

The "addition to itself" naturally has to be conceived of as repeated any number of times; the meaning therefore is: multiplied by a natural number.

There are great differences between the translations given by various editors of the last words of this postulate *viz.* παντὸς τοῦ προτεθέντος τῶν πρὸς ἄλληλα λεγομένων. Without laying claim to completeness, we mention the following versions:

a) *Editio Princeps*[1]: *omnem propositam sui generis quantitatem.*
b) *Heiberg*[2]) : *quamvis magnitudinem datam earum, quae cum ea comparari possint.*
c) *Heath*[3]) : *any assigned magnitude among those which are comparable with [it and with] one another.*
d) *Heath*[4]) : *any assigned magnitude of the same kind.*
e) *Ver Eecke*[5]) : *toute grandeur donnée ayant un rapport avec l'une et l'autre des premières.*
f) *Czwalina*[6]) : *jede der beiden gegebenen Grössen.*

When therefore the two unequal lines (surfaces, solids) of which Archimedes speaks are called A and B respectively, and it is assumed that $A-B=C$, the translations a), b), and as far as the addition between brackets is concerned also c) state that there exists a number n such that $n\cdot C>D$ when D is comparable with C or "of the same kind"; in d) also this seems to be suggested. This naturally raises the question when two magnitudes are called of

[1]) Quoted on p. 41, Note 3; p. 2 of the Latin translation.
[2]) *Opera* I, 9, and slightly differently in the mention of the *axioma* in **Spir.** (*Opera* II, 13): ...*earum, quae inter se comparari possint.*
[3]) Heath, *Archimedes*, p. 4.
[4]) Heath, *Greek Mathematics* II, 35.
[5]) Ver Eecke, *Archimède*, p. 6.
[6]) Czwalina, *Kugel und Zylinder*, p. 9.

the same kind or comparable. The answer to this is to be inferred from the Definitions 3 and 4 of Euclid V, which state, when correlated with each other, that two magnitudes are of the same kind (or: have a ratio to one another) when they are capable, upon multiplication, of exceeding one another. In the translations a)–d) it is therefore said that the magnitude C is of the same kind with any magnitude D with which it is of the same kind.

The Greek text, however, contains quite a different statement. It does not refer to *comparable* magnitudes, but to magnitudes *compared* with each other, and by this only the magnitudes A and B can be meant. The postulate therefore states that when C is the difference between two lines (surfaces, solids), the number n may be so chosen that $n\cdot C$ is greater than any line (surface, solid) whatever.

It is thus clear that the translation e), though philologically not exactly in accordance with the text, renders at least the meaning of the statement correctly. In fact, according to the theory of proportions D only then has a ratio to A and to B when D is of the same kind with A and with B.

The translation f), finally, is not at all in agreement with the text.

The view adopted by us above and expressed in the translation is to be found, among others, in the following translations:

Mersenne[1]): *quamcumque dictarum, et inter se collatarum magnitudinum.*
Nizze[2]) : *jede gegebene Grösse von der Art der verglichenen.*

We have gone into the matter of the correct translation of the fifth postulate in some detail because it is only after the exact meaning of Archimedes' words has been established that a discussion is possible of the generally neglected, but no less urgent question what motive may have induced him to include this assumption. Indeed, if one of the translations a)–d) is adopted, it is incomprehensible what he may have intended by it. In that case it can only be interpreted as the formulation of Definition 4 of Euclid

[1]) *Universae Geometriae, Mixtaeque Mathematicae Synopsis, Et Bini Refractionum Demonstratarum Tractatus. Studio et Opera* F. M. Mersenne M. Parisiis MDCXLIV.
[2]) Quoted on p. 43, note 2.

of straight line segments. Owing to this, among other things, the curves considered by Archimedes may also partially coincide with the straight line determined by the extremities.

II. *I call such a line "concave in the same direction"* (ἐπὶ τὰ αὐτὰ κοίλη) *when it has the property that, if any two points on it are taken, either all the straight lines*[1]) *connecting such points fall on the same side of the line or some fall on one and the same side, while others fall on the line itself, but none on the other side.*

The lines defined in Axiom II therefore belong to the category circumscribed by Axiom I; in most translations this does not emerge sufficiently[2]). In modern terminology it is stated by Axiom II that a line which is concave on the same side includes a convex surface with the line segment joining its extremities.

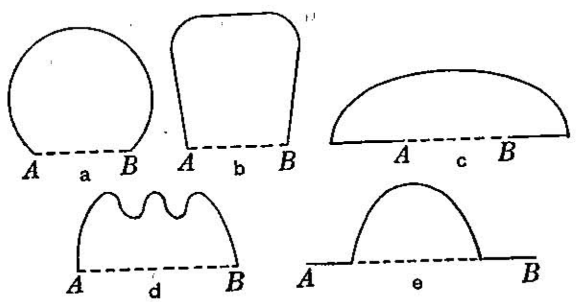

Fig. 51.

Of the lines shown in Fig. 51 with the extremities A and B, a, b, and c are of the type described, d and e are not.

III. *Similarly also there are certain terminated surfaces*[3]) *which are not themselves in a plane, but have their boundaries in a plane, and which either lie wholly on the same side of the plane containing the boundaries or have no part of them on the other side thereof.*

Here again the meaning is that the surfaces under consideration may also consist of one or more plane faces.

[1]) The Greeks do not distinguish between a straight line and a straight line segment. In the translations, however, we shall use the latter term wherever this is required for the sake of clarity. In most cases, moreover, the meaning is easily inferred from the context.

[2]) Eutocius states it explicitly (*Opera* III, 4; line 16): ἐκ δὲ τούτων ἦν ἡ ἐπιλογὴ τῶν ἐπὶ τὰ αὐτὰ κοίλων. An erroneous translation is to be found in Czwalina, *Kugel und Zylinder*, p. 8.

[3]) By this is meant that the surface under consideration has a boundary.

IV. *I call such surfaces "concave in the same direction" when they have the property that, if any two points on them are taken, the straight lines connecting such points either all fall on the same side of the surface, or some fall on one and the same side, others upon the surface itself*[1]), *but none on the other side.*

V. *When a sphere is cut by a cone which has its vertex at the centre of the sphere, I use the term solid sector* (τομεύς στερεός) *for the figure comprehended by the surface of the cone and the surface of the sphere included within the cone.*

VI. *When two cones with the same base have their vertices on opposite sides of the plane of the base, so that their axes lie in one straight line, I use the term solid rhombus* (ῥόμβος στερεός) *for the solid figure made up of the two cones.*

3. *Lambanomena.*

By this title is denoted a group of postulates or axioms in the sense of "unproved fundamental propositions on known figures".

I assume the following:

I. *That of the lines which have the same extremities the straight line is the least.*

II. *That of the other lines*[2]), *if, lying in one plane, they have the same extremities, two are unequal whenever both are concave in the same direction and moreover one of them is either wholly included between the other and the straight line which has the same extremities with it, or is partly included by and partly coincides with the other; and that the line which is included is the lesser.*

III. *Similarly, that of the surfaces which have the same boundaries, if the latter are in one plane, the plane surface is the least.*

IV. *That of the other surfaces which have the same boundaries, if the boundaries are in one plane, two are unequal whenever both are concave in the same direction and moreover one of them is either wholly included between the other and the plane which has the same boundaries, or is partly included by and partly coincides with the other; and that the surface which is included is the lesser.*

[1]) Here we have to think either of a ruled surface (in this case a cone or a cylinder) or of a surface having one or more plane faces.

[2]) *i.e.* of all lines which are not straight (*i.e.* the καμπύλαι γραμμαί).

itself half as large again as the sphere[1]), and its surface is half as large again as the surface of the sphere[2]).

This preliminary and partial summary of the contents of the treatise is followed by a statement the exact purport of which is doubtful, and which we will therefore reproduce in the original, with a specimen translation opposite:

ταῦτα δὲ τὰ συμπτώματα τῇ φύσει προυπῆρχεν περὶ τὰ εἰρημένα σχήματα, ἠγνοεῖτο δὲ ὑπὸ τῶν πρὸ ἡμῶν περὶ γεωμετρίαν ἀνεστραμμένων οὐδενὸς αὐτῶν ἐπινενοηκότος, ὅτι τούτων τῶν σχημάτων ἐστὶν συμμετρία· διόπερ οὐκ ἂν ὀκνήσαιμι ἀντιπαραβαλεῖν αὐτὰ πρός τε τὰ τοῖς ἄλλοις γεωμέτραις τεθεωρημένα καὶ πρὸς τὰ δόξαντα πολὺ ὑπερέχειν τῶν ὑπὸ Εὐδόξου περὶ τὰ στερεὰ θεωρηθέντων, ὅτι πᾶσα πυραμὶς τρίτον ἐστὶ μέρος πρίσματος τοῦ βάσιν ἔχοντος τὴν αὐτὴν τῇ πυραμίδι καὶ ὕψος ἴσον, καὶ ὅτι πᾶς κῶνος τρίτον μέρος ἐστὶν τοῦ κυλίνδρου τοῦ βάσιν ἔχοντος τὴν αὐτὴν τῷ κώνῳ καὶ ὕψος ἴσον· καὶ γὰρ τούτων προυπαρχόντων φυσικῶς περὶ ταῦτα τὰ σχήματα, πολλῶν πρὸ Εὐδόξου γεγενημένων ἀξίων λόγου γεωμετρῶν συνέβαινεν ὑπὸ πάντων ἀγνοεῖσθαι μηδ' ὑφ ἑνὸς κατανοηθῆναι.	These properties were all along naturally inherent already in the figures referred to, but they were unknown to those who were before our time engaged in the study of geometry, because none of them realized that there exists symmetry between these figures[3]). Therefore I would not hesitate to compare these properties with the insight gained by other geometers and with those of the theorems of Eudoxus on solid figures which to my mind are the most excellent, *viz.* that any pyramid is one-third of the prism which has the same base as the pyramid and equal height, and that any cone is one-third of the cylinder which has the same base as the cone and equal height[4]). For, though these properties also were already naturally inherent in the figures all along, and there were many important geometers before Eudoxus, they have remained unknown to them all and have not been realized by anyone[1]).

[1]) Greek has no word for volume. This magnitude is denoted by the name of the figure itself. The same is usually done in planimetrical discussions with regard to the area of a figure.

[2]) S.C. I, 34. Corollary.

[3]) We are here using the word symmetry in the sense of commensurability, which is to be attributed to it both etymologically and historically.

[4]) Euclid XII, 7, Porism. XII, 10.

If the suggested translation is correct, Archimedes here seems to voice his astonishment that geometrical figures may have remarkable properties inherent in them, *i.e.* without their being stated in the definition we give of them, which properties may long remain unnoticed, in spite of their simplicity. It is the typical mathematician's astonishment at the unsuspected intrinsic wealth of his own definition which is being expressed here.

At the close of his preface the author presents his work to all experts, inviting their verdict; he regrets that it could not be published while Conon was still alive, because the latter would have been eminently capable of grasping its contents and giving his opinion on them. There now follow first the *axiomata* and the *lambanomena* (assumptions) on which the treatise is to be based.

2. *Axiomata.*

The first group of the fundamental propositions has been rightly termed *Axiomata* in so far as they postulate the existence of certain types of curves and surfaces. On the other hand this group also contains two purely nominal definitions.

I. *There are in a plane certain terminated bent lines*[2]) *which either lie wholly on the same side of the straight line joining their extremities or have no part of them on the other side thereof.*

Eutocius comments on this[3]) that the category of the bent lines (*καμπύλαι γραμμαί*) also includes lines which consist wholly or partly

[1]) Elsewhere (introduction to the *Method, Opera* II, 430) Archimedes says that a not inconsiderable part of the merit of these propositions is due to Democritus, who was the first to enunciate them, though without giving any proof (*i.e.* without giving any perfectly exact proof). These two statements are therefore at variance with each other, a fact which is not to be removed by supposing that during his writing of **S.C.** Archimedes did not yet know the things told by him in **Meth.**; for in view of various other reasons the *Method* is assumed to be the earlier of the two treatises.

[2]) By this is meant that the part of the curve which is being considered has two extremities.

[3]) *Opera* III, 4; lines 8 *et seq.*

The propositions 9.4 and 9.5 finally state that

$$\lim_{B \to \Gamma} \frac{BE}{I\Gamma} = \operatorname{tg}\alpha \,.$$

10. *Elements of Mechanics.*

10. For the treatise *On Spirals* Archimedes requires a couple of lemmas on uniform motion.

10.1. **Spir.** 1. Here it is stated that two distances described by a point in the same uniform motion are proportional to the times of describing them.

Let the motion take place on AB. On this consider two distances $\Gamma\Delta$ and ΔE; represent the times of describing them successively by ZH, $H\Theta$. It has now to be proved that

$$(\Gamma\Delta, \Delta E) = (ZH, H\Theta) \,.$$

The proof is based directly on the Euclidean definition of proportion. In fact, let $A\Delta$ be a multiple of $\Gamma\Delta$, ΔB a multiple of ΔE, and let $A\Delta > \Delta B$ (this is possible on the ground of the axiom of Eudoxus, which is formulated once more as a lemma in the introduction to *On Spirals*).

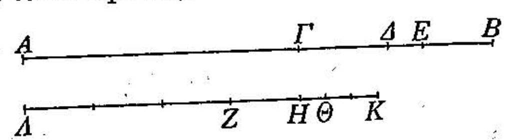

Fig. 50.

Now let ΛH be the same multiple of ZH as $A\Delta$ is of $\Gamma\Delta$, KH the same multiple of ΘH as ΔB of ΔE. Since the moving point needs for all the distances equal to $\Gamma\Delta$ a time equal to ZH, it appears that ΛH stands for the time needed for $A\Delta$, and likewise HK for that required for ΔB. From $A\Delta > \Delta B$ it then follows that $\Lambda H > HK$. In the same way it is realized that from the inequality of any multiples of ZH, $H\Theta$ follows the inequality in the same sense of equal multiples of $\Gamma\Delta$, ΔE, from which the proportion of the proposition is concluded to be true.

As basis of the argument is apparently used a definition (not stated) of uniform motion as motion in which equal distances are

described in equal times. It may seem that the proposition follows at once from this, and that Archimedes already makes use of it in the proof. In this way, however, the proposition can only be understood for the case of two distances being in a rational ratio to each other; the proof given serves to demonstrate its validity for irrational ratios of distances as well.

10.2. **Spir.** 2. In this it is shown that two distances described in uniform motion in certain times are proportional to the distances described in a different uniform motion in the same times. This follows at once from 10.1.

CHAPTER IV.

ON THE SPHERE AND CYLINDER

BOOK I.

1. *Introduction.*

The first of the two books into which the treatise *On the Sphere and Cylinder* is divided opens with a letter to Dositheus, in which Archimedes reminds him that on a former occasion he already sent him the proof of the proposition that any segment of an orthotome is four-thirds of the triangle with the same base and equal height[1]). He is now going to demonstrate new propositions:

a) The surface of any sphere is four times its greatest circle[2]).

b) The surface of any segment of a sphere is equal to that of a circle whose radius is equal to the straight line drawn from the vertex of the segment to any point of the circumference of the circle which is the base of the segment[3]).

c) Any cylinder[4]), whose base is equal to the greatest circle of those in the sphere and whose height is equal to the diameter, is

[1]) **Q.P.** 17 and 24. *Vide* Chapter X.

[2]) **S.C. I**, 33. The Greeks say "greatest circle" (μέγιστος κύκλος), which is more logical than our "great circle".

[3]) **S.C. I**, 42, 43.

[4]) By a cylinder in **S.C.** is meant a right circular cylinder, by a cone a right circular cone.