Name:

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| MWF 11 Sander Kupers |
| MWF 12 Hakim Walker |
| TTH 10 Ana Balibanu |
| TTH 10 Morgan Opie |
| TTH 10 Rosalie Belanger-Rioux |
| TTH 11:30 Philip Engel |
| TTH 11:30 Alison Miller |

- Please fill in your name and mark your section.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total: | | 110 |

Problem 1) (20 points) True or False? No justifications are needed.

1) T **F** The geometric multiplicity of the eigenvalue 1 of the shear $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is equal to 2.

**Solution:**

2) **T** F Two symmetric $4 \times 4$ matrices are similar if their trace, determinant and eigenvalues are the same.

**Solution:**
They both can be diagonalized.

3) T **F** Every orthogonal $3 \times 3$ matrix satisfies $A = A^{-1}$.

**Solution:**
The definition is $A^T A = I_n$.

4) **T** F If $A$ is a $5 \times 5$ matrix and $A^2 = I_5$ and $A = A^T$, then $A$ is similar to an orthogonal matrix.

**Solution:**
We have then $AA^T = I_5$. Its not only that $A$ is similar to an orthogonal matrix, it is an orthogonal matrix.

5) T **F** The trace of a $4 \times 4$ matrix $A$ does not change under row reduction.

**Solution:**
Already a scaling does change the eigenvalues.

6) **T** F Every $3 \times 3$ matrix $A$ for which $T(x) = Ax$ is a reflection about a plane can be diagonalized.

**Solution:**
It is symmetric

7) T **F** The recursion $x_{t+1} = x_t + 3x_{t-1} + 1$ can be written as a vector equation $\vec{v}_{t+1} = A\vec{v}_t$ for a $2 \times 2$ matrix $A$ and vector $\vec{v}_t = [x_t, x_{t-1}]^T$.

**Solution:**
It is affine but not linear.

8) **T** F The determinant of a reflection matrix is either 1 or $-1$.

**Solution:**
Yes, it is an orthogonal matrix

9) T **F** The nullity of a $n \times n$ matrix $A$ is the same as the rank of $A^T$.

**Solution:**
The nullity of $A$ is the same than the nullity of $A^T$.

10) **T** F For any $3 \times 3$ matrix, we have $\det(A^5) = \det(A)^5$.

**Solution:**
From the product formula

11) **T** F A discrete dynamical system $\vec{v}_{t+1} = A\vec{v}_t$ defined by a $2 \times 2$ matrix $A$ is asymptotically stable if $A^8 = 0$.

**Solution:**
By definition.

12) **T** F A $2 \times 2$ matrix for a reflection about a line always has trace 0.

**Solution:**
The diagonal entries add up to zero.

13) **T** F The matrix $A = \begin{bmatrix} 4 & 0 \\ -2 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 3 & 3 \\ 0 & 4 \end{bmatrix}$.

**Solution:**
Because both have the same trace and determinant, their eigenvalues are the same. Because the eigenvalues are different they can both be diagonalized to the same diagonal matrix.

14) | T | F | If $A$ is a $5 \times 3$ matrix of rank 3, then the least square solution of $Ax = b$ is unique.

**Solution:**
Yes, this implies that the kernel of $A$ is trivial.

15) | T | F | If $A$ is asymptotically stable, then $A^T$ is asymptotically stable.

**Solution:**
$A$ and $A^T$ have the same eigenvalues

16) | T | F | If $A$ is similar to $C$ and $B$ is similar to $D$, then $A + B$ is similar to $C + D$.

**Solution:**
$\text{Diag}(0, 1)$ is similar to $\text{Diag}(1, 0)$ and $\text{Diag}(1, 0)$ is similar to $\text{Diag}(1, 0)$ but $\text{Diag}(1, 1)$ is not similar to $\text{Diag}(2, 0)$.

17) | T | F | If $A$ is similar to $A^{-1}$ then $A$ is the identity.

**Solution:**
This is not true for the shear.

18) | T | F | The sum of the geometric multiplicities of a $n \times n$ matrix is always $n$.

**Solution:**
Take the shear matrix appearing in the first TF problem. It has geometric multiplicity 1.

19) | T | F | If $A$ has the same determinant as $B^2$, then $B$ has the same determinant as $A^2$.

**Solution:**
Its already not true for $A = 4I_n$ and $B = 2I_n$.

20) T F If $A$ is a non-invertible $3 \times 3$ matrix with trace 6 and such that 5 is an eigenvalue, then 1 is another eigenvalue of $A$.

**Solution:**
The sum of the eigenvalues is the trace.

Total

Problem 2) (10 points) No justifications are needed.

a) (2 points) Which matrices have the property that the system $x(t+1) = Ax(t)$ is asymptotically stable (for which we just write "stable")?

| Matrix | stable | not stable |
|---|---|---|
| $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ | | |
| $\begin{bmatrix} 0.9 & 1 \\ 0 & 0.9 \end{bmatrix}$ | | |
| $\begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$ | | |
| $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ | | |

**Solution:**
this is not stable: the eigenvalues are $0, 1$.
This is stable: the eigenvalues are $0.9, 0.9$.
this is stable: this is a rotation dilation matrix with eigenvalues $1/3 \pm 1/3i$.
this is not stable: the determinant is 1, so that one of the eigenvalues is in absolute value $\geq 1$.

b) (2 points) Which identities hold for a $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with eigenvalues $\alpha, \beta$.

| Identity | always true | not always true |
|---|---|---|
| $\mathrm{tr}(A) = \alpha + \beta$ | | |
| $\det(A) = \alpha\beta$ | | |
| $\det(A) = \mathrm{tr}(A)$ | | |
| $\det(A - \lambda I_2) = (\alpha - \lambda)(\beta - \lambda)$ | | |

**Solution:**
Only the identity $\det(A) = \mathrm{tr}(A)$ is not always true.

c) (2 points) Which of the following complex numbers are real? Remember that we defined $w^z = e^{z \log(w)}$ and $\log(z) = \ln|z| + i \arg(z)$ with $0 \leq \arg(z) < 2\pi$ for any complex numbers $w \neq 0, z \neq 0$ and where $\arg(z)$ is the angle so that $z = |z|e^{i \arg(z)}$.

| Number | is real | is not real |
|---|---|---|
| $1^i$ | | |
| $2^i$ | | |
| $i^2$ | | |
| $i^i$ | | |

**Solution:**
Only $2^i$ is not real. We can calculate $2^i = e^{i \log(2)} = \cos(\log(2)) + i \sin(\log(2))$.

d) (2 points) Which types of matrices are always diagonalizable over the real numbers?

| Type of matrix | diagonalizable | not diagonalizable |
|---|---|---|
| orthogonal | | |
| projection | | |
| symmetric matrix | | |
| all geometric multiplicities are 1 | | |

**Solution:**
The projection matrix and any symmetric matrix is diagonalizable. An orthogonal matrix is only diagonalizable over the complex. The shear is an example of a matrix for which all geometric multiplicities is 1 but it is not diagonalizable.

e) (2 points) If $S$ is a $3 \times 3$ matrix whose columns are given by an eigenbasis of a matrix $A$ which has eigenvalues $1, 2, 3$, then

| Statement | is always true | is not always true |
|---|---|---|
| $S$ is invertible | | |
| $S$ is a projection | | |
| $S$ is orthogonal | | |
| $S$ is symmetric | | |

**Solution:**
Only the statement $S$ is invertible is always true. The matrix $S$ can be pretty anything. Start with a diagonal matrix $B$ take an invertible but otherwise arbitrary $S$, then define $A = SBS^{-1}$. The eigenbasis of $A$ is then given by the columns of $S$.

Problem 3) (10 points) No justifications are needed

a) (2 points) One of the following formulas gives the projection onto the image of $A$. Which one?

| | |
|---|---|
| $A(A^TA)^{-1}A^T$ | |
| $A^T(A^TA)^{-1}A$ | |
| $A(AA^T)^{-1}A^T$ | |
| $A^T(AA^T)^{-1}A^T$ | |
| $(AA^T)^{-1}A^T$ | |
| $(A^TA)^{-1}A^T$ | |

**Solution:**
This was a knowledge question (or if not known by heart obtained by the geometric condition that $Ax - b$ is perpendicular to the image, so that it is in the kernel of $A^T$ and so $A^T(Ax - b) = 0$ which gives $x = (A^TA)^{-1}A^Tb$ and $Ax = A(A^TA)^{-1}A^Tb$. The first is the right one.

b) (2 points) Which of the following matrices have real eigenvalues?

| | | |
|---|---|---|
| $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ | | |
| $A = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$ | | |
| $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ | | |
| $A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ | | |

**Solution:**
All have real eigenvalues except the first one. The first one is a rotation dilation matrix. The others are either symmetric or triangular.

c) (2 points) For which of the following matrices does the identity matrix appear either for $Q$ or for $R$ in the $QR$ factorization?

| | | |
|---|---|---|
| $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ | | |
| $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | | |
| $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ | | |
| $A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ | | |

**Solution:**
In the first two cases, we have either an upper triangular or orthogonal matrix so that the QR decomposition is trivial.

d) (2 points) Which of the following matrices is useful to find the closed form solution for the recursion $x_{t+1} = 5x_t - 4x_{t-1}$?

| | | | |
|---|---|---|---|
| $A =$ | $\begin{bmatrix} 5 & -4 \\ 1 & 0 \end{bmatrix}$ | | |
| $A =$ | $\begin{bmatrix} 5 & -4 \\ 0 & 1 \end{bmatrix}$ | | |
| $A =$ | $\begin{bmatrix} 5 & 0 \\ -1 & 0 \end{bmatrix}$ | | |
| $A =$ | $\begin{bmatrix} 5 & 0 \\ -1 & 4 \end{bmatrix}$ | | |

**Solution:**
The first one. Write $x_{t+1} = 5x_t - 4x_{t-1}, x_t = x_t$, then get the matrix.

e) (2 points) Two of the following statements are part of the spectral theorem. Which ones?

| | |
|---|---|
| A symmetric matrix has an orthonormal eigenbasis | |
| An orthogonal matrix is diagonalizable over the reals | |
| A real matrix has real eigenvalues | |
| A symmetric matrix has real eigenvalues | |
| A matrix with distinct eigenvalues has an orthonormal eigenbasis | |

**Solution:**
The first and the fourth one constitute the spectral theorem.

Problem 4) (10 points)

Find the function $y = ax + bx^3$ which provides the best fit for the data

| x | y |
|---|---|
| 1 | 2 |
| -2 | 1 |
| -1 | 2 |
| 0 | 2 |

.

**Solution:**
Write the data as a system of equations $Ax = b$ with $A = \begin{bmatrix} 1 & 1 \\ -2 & -8 \\ -1 & -1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}$.
The rest is routine. We have $x = (A^T A)^{-1} A^T b = [-1, -1]^T/6$. A few check points: $(A^T A) = \begin{bmatrix} 6 & 18 \\ 18 & 66 \end{bmatrix}$. $A^T b = [-2, -8]^T$. The best fit is $x/6 - x^3/6$.

Problem 5) (10 points)

The recursion $x_{t+1} = 3x_t - 2x_{t-1}$ with $x_0 = 0, x_1 = 1$ is an example of a **Lucas sequence**. It starts with $0, 1, 3, 7, 15, 31, 63, ....$ Recall that we can write the problem as a discrete dynamical system

$$\vec{v}_{t+1} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \vec{v}_t \ .$$

Find a closed form solution for $\vec{v}_t$ if the initial condition is

$$\vec{v}_0 = \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ .$$

**Isabel Vogt** and **Jesse Silliman** recently showed that the given Lucas sequence contains no perfect powers like for example $7^3$. Isabel has been a Harvard math concentrator here. The preprint is here http://arxiv.org/pdf/1307.5078v2.pdf

**Solution:**
The eigenvalues are $1, 2$ with eigenvalues $[1, 1]^T$ and $[2, 1]^T$. We write $[1, 0]^T = [2, 1] - [1, 1]$ so that the closed form solution is

$$(-1)1^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We can write this as $x(t) = 2^t - 1$.

Problem 6) (10 points)

a) (5 points) We look at the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix} \ .$$

Find all the eigenvectors and the characteristic polynomial $f_A$ of $A$.

b) (3 points) You know that $A$ is similar to a diagonal matrix $B$. Write down this matrix $B$.

c) (2 points) Also write down the matrix $S$ which satisfies $B = S^{-1}AS$.

**Solution:**
a) There is an eigenvalue 5 (sum of rows), an eigenvalue 0 (first and last row are parallel) and an eigenvalue 2 (trace). The characteristic polynomial is

$$(-\lambda)(5-\lambda)(2-\lambda)\ .$$

The eigenvectors are $[1,1,-4]^T$,$[1,1,1]^T$ and $[1,-2,1]^T$.
b) The matrix is $\text{Diag}(0,5,2)$.
c) The matrix $S$ contains the eigenvectors as columns:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 4 & 1 & 1 \end{bmatrix}\ .$$

Problem 7) (10 points)

a) (5 points) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 3 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}\ .$$

Note that it is of the form $Q + 3Q^{-1} + 2I_7$, where $Q$ is a matrix of a type you have seen a lot.

b) (5 points) Determine all the eigenvectors of that matrix using $A = Q + 3Q^{-1} + 2I_7$.

Remark: You do not have to simplify the expressions you get for the eigenvalues and eigenvectors.

**Solution:**
a) The eigenvalues of $Q$ are $\lambda_k = e^{2\pi i k/7}$ with eigenvectors

$$v_k = \begin{bmatrix} \lambda_k^6 \\ \lambda_k^5 \\ \lambda_k^4 \\ \lambda_k^3 \\ \lambda_k^2 \\ \lambda_k \\ 1 \end{bmatrix} .$$

The eigenvalues of $A$ are $e^{2\pi ik/7} + 3e^{-2\pi ik/7} + 2$.
b) The eigenvectors of $A$ are the same than the eigenvalues of $Q$. Note that we do not fill in the eigenvalues of $A$ but keep the same vector.

Problem 8) (10 points)

a) (2 points) What is the determinant of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 4 & 5 \\ 1 & 2 & 4 & 4 & 5 \\ 1 & 2 & 3 & 5 & 5 \\ 1 & 2 & 3 & 4 & 6 \end{bmatrix}$?

b) (2 points) Find the determinant of the matrix $\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix}$.

c) (3 points) To commemorate the "Go" match between computer and human from March 2016, we pose a determinant problem for a $19 \times 19$ matrix: evaluate

$$\begin{bmatrix}
\boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \boxed{1} & 0 & \boxed{1} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 0 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & \boxed{1} & \boxed{1} & \boxed{1} & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0
\end{bmatrix}$$

d) (3 points) find the determinant of $\begin{bmatrix} 7 & 2 & 2 & 2 & 2 & 2 \\ 2 & 7 & 2 & 2 & 2 & 2 \\ 2 & 2 & 7 & 2 & 2 & 2 \\ 2 & 2 & 2 & 7 & 2 & 2 \\ 2 & 2 & 2 & 2 & 7 & 2 \\ 2 & 2 & 2 & 2 & 2 & 7 \end{bmatrix}$.

**Solution:**
a) Row reduce (subtract first row from all others gives triangular matrix) 1.
b) Partitioned matrix with 3 blocks $A, B, C$. The determinant is the product $3^7$.
c) See one pattern with one up crossing or swap last two rows to get a triangular matrix. The answer is $-1$.
d) Subtract 5 to get a matrix with 5 zero eigenvalues and eigenvalue 12. The eigenvalues of the matrix are $5, 5, 5, 5, 5, 17$. The determinant is $5^5 \cdot 17$.

Problem 9) (10 points)

a) (4 points) Use Gram-Schmidt to find an orthonormal basis of the image of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

b) (3 points) Let $Q$ be the matrix which contains this orthonormal basis. What is the $2 \times 2$ matrix $Q^T Q$?

c) (3 points) What is the projection $QQ^T \vec{v}$ on the image of $A$ if $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.

**Solution:**
a) First scale the first column to get $u_1 = [1,1,1,1]^T/2$. Then build $w_2 = v_2 - (u_1 \cdot v_2)u_1$, then normalize. This gives $u_2 = [3,-1,-5,-3]^T/\sqrt{44}$. The new basis contains $u_1, u_2$.
b) The matrix is $Q^T Q = I_2$ because the columns are all orthonormal.
c) Compute $QQ^T b = [3,-1,-5,3]/11$.

Problem 10) (10 points)

The matrix

$$D = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

has the property that

$$L = D^2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} .$$

a) (4 points) Find the eigenvalues and eigenvectors of $L$.
b) (2 points) Now find the eigenvalues of $D$.
c) (2 points) The **Pseudo determinant** of a matrix is the product of the nonzero eigenvalues of $A$. Find the pseudo determinant of the matrix $L$.

Pseudo determinants are important in network theory: they count the number of rooted trees in a network. In the example just given, the network has four nodes and all are connected. P.S. You have here computed the Dirac operator $D$ and the Laplacian of a network with two nodes connected by an edge.

**Solution:**
a) We can compute the characteristic polynomial and find the roots $0, 2, 2$ or then see geometrically in the $xy$ plane we have eigenvalues $0, 2$ and in the $z$ direction an eigenvalue 2. The eigenvectors $[1,1,0]^T, [1,-1,0]^T, [0,0,1]^T$.
b) The eigenvalues are $0, \sqrt{2}, -\sqrt{2}$.
c) The product of the nonzero eigenvalues is 4.