Name:

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MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
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MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible to the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T **F** Any 88×77 matrix A has exactly 88 columns.

Solution:
It has 77 columns.

2) **T** F If a system of equations $Ax = b$ has infinitely many solutions, then A has a non-trivial kernel.

Solution:
Yes, then $Ax = 0, Ay = 0$ gives $A(x - y) = 0$.

3) **T** F Given a basis $\mathcal{B} = \{v_1, v_2\}$ of a 2-dimensional space V, then $\{v_1 - v_2, v_2\}$ is a basis in V.

4) T **F** For any two invertible 2×2 matrices A and B we have $(AB)^{-2} = A^{-2}B^{-2}$. (The notation A^{-2} means $(A^{-1})^2$.)

5) **T** F If V is a subspace of R^3 and W is the orthogonal complement of V, then $\dim(V) + \dim(W) = 3$.

6) T **F** If A is a matrix for which $A^3 = 0$, then A^2 must be the zero matrix.

7) **T** F If a matrix A satisfies $A^2 = 0$ and v is in the image of A then v is in the kernel of A.

8) T **F** The set of smooth functions satisfying $f''(0)f(0) = 0$ is a linear space.

9) **T** F If A is a 3×5 matrix then the equation $Ax = 0$ has infinitely many solutions.

Solution:
The kernel is not trivial

10) **T** F If A is a non-zero 1×4 matrix then $Ax = 3$ has at least one solution.

Solution:
The solution space is one dimensional

11) T **F** The projection from the plane $V = \mathbf{R}^2$ onto the line $x = y$ in V is invertible.

12) **T** F If a matrix A has rank 4, then for any invertible matrix B, the matrix AB has rank 4.

13) T **F** If A and B are invertible 2×2 matrices, then $A^2 + B^2$ is invertible.

Solution:
Take A with $A^2 = I$, and B with $B^2 = -1$.

14) **T** F Given two 2×2 reflection matrices A, B then $(AB)^{-1} = BA$.

Solution:
Yes, we have to change the order

15) T **F** If A, B are 2×2 matrices and $A^2 = B^2$, then $A = B$ or $A = -B$.

Solution:
Take any two reflections.

16) **T** F Given a 4×4 matrix A, then the kernel of A^3 contains the kernel of A.

Solution:
You have done that in a homework.

17) **T** F If A is a 3×3 matrix and A is invertible, then $\text{rref}(A)$ is invertible.

Solution:
The kernel does not change.

18) **T** F There exists a 3×3 matrix A such that $A^{2049} = I_3$ and A is not equal to I_3.

Solution:
Take a rotation about a line with angle 360/2049 degrees.

19) | T | **F** | If A is a vertical shear and B is a horizontal shear then AB is a rotation.

Solution:
Take A to be the identity.

20) | **T** | F | If A and B are invertible 2×2 matrices and $BAB^{-1} = I_2$, then $A = I_2$.

[ ] Total

Problem 2) (10 points) No justifications are needed.

In all sub problems a-d), each mismatch is one point off until all points are depleted.

a) (3 points) Decide in each case whether the matrix is in row reduced echelon form.

Matrix	is row reduced	is not row reduced
$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		
$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$		

Solution:

Only the first and last matrix is row reduced.

b) (2 points) Mark the linear spaces. Remember that C^∞ denotes the set of smooth functions on the real line.

Check if linear space	Space
	The set of polynomials $f(x) = a + bx + cx^2$ in P_2 with $a \geq 0, b \geq 0, c \geq 0$
	The set of functions f in C^∞ for which $f'(1) + f(2) = 0$.
	The set of 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for which $a^2 + b^2 = 0$.

Solution:

Only the second one.

c) (3 points) Check all the boxes which apply:

Matrix S	S is invertible	S is a rotation	$\mathcal{B} = \{Se_1, Se_2, Se_3\}$ is a basis.
$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$			
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$			
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$			
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$			

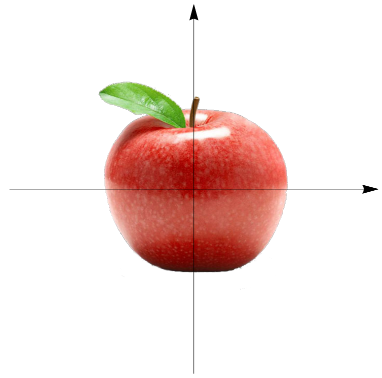
d) (2 points) Match the transformation type: A: **rotation dilation**, B: **reflection dilation**, C: **shear dilation**, D: **projection dilation**

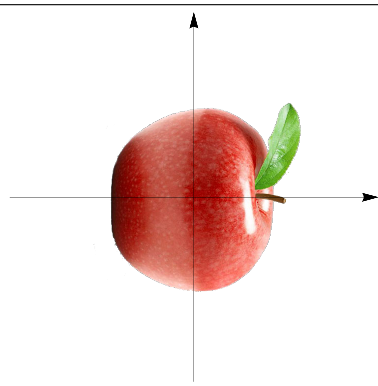
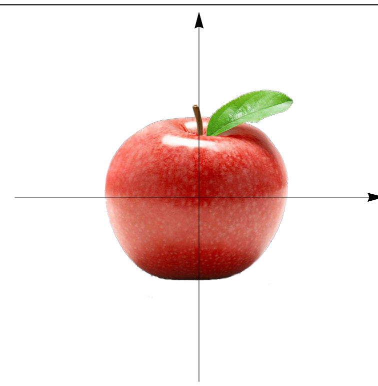
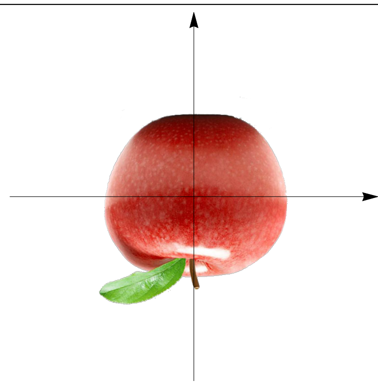
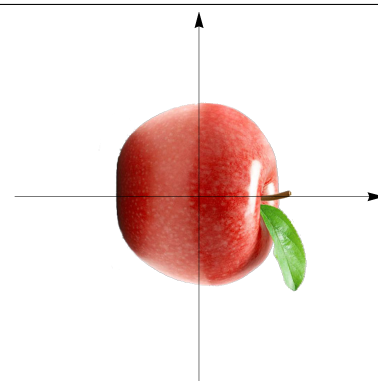
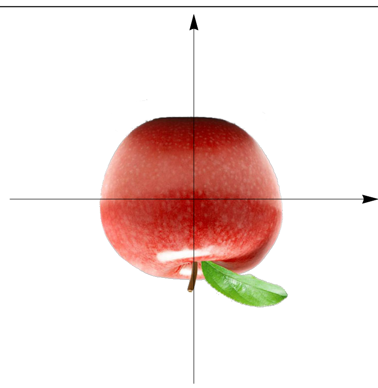
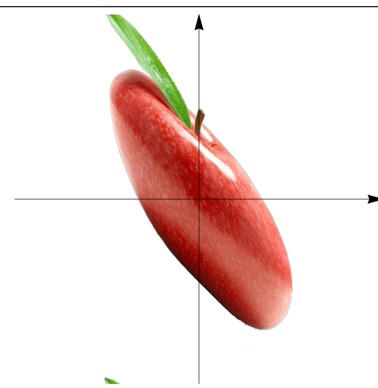
Fill A-D				
Matrix	$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & -4 \\ 4 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ 4 & -4 \end{bmatrix}$	$\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

Solution:
a) Row reduced
Not row reduced
Row reduced
b) No, Yes, Yes
c) Yes, yes yes
yes, no yes
no, no no
yes yes yes
d) DABC

Problem 3) (10 points) No justifications are necessary.

"An apple a day keeps the doctor away" is a common English-language proverb of Welsh origin. Match the pictures with the transformations. For the grading: each mismatch takes 2 points off until all 10 points are depleted.



A-F		A-F	
			
			
			

A=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ B=$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ C=$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ D=$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ E=$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ F=$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Solution:
D, C
E, A
F, B

Problem 4) (10 points)

Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 10 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ x_1 + 2x_2 &= 3\,. \end{aligned}$$

a) (2 points) Give the matrix A and the vector b such that this system reads as $Ax = b$. Also write down the augmented matrix.

b) (5 points) Find all the solutions to the system $Ax = b$.

c) (3 points) Is there a vector c so that the system $Ax = c$ has no solution? If yes, find one.

Solution:
a) The row reduced echelon form is

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

b) $[3,0,1,1]^T + s[-2,1,0,0]^t$.

Problem 5) (10 points)

Let us perform some matrix algebra with the **Harvard and Yale matrices**

$$H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

a) (3 points) Compute the Harvard-Yale commutation matrix $HY - YH$. (Your computation shows that Harvard and Yale do not commute).

b) (3 points) To figure out which institution is more important, compute H^2 and Y^2. Which of the two squares has the largest entry?

c) (2 points) Compute $H \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Then compute $Y \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

d) (2 points) What is $H^2 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$? Do you see the pattern? What is $H^5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$?

Solution:

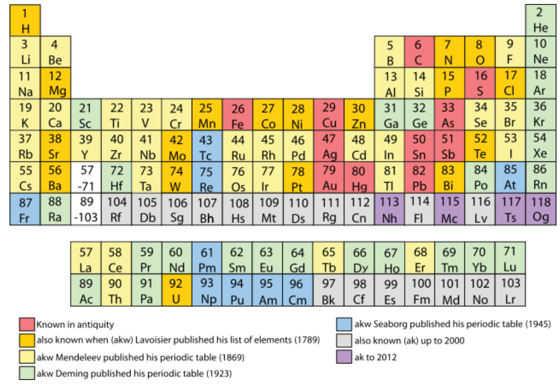
a) $HY - YGH = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

b) $H^2 = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 1 & 3 \\ 3 & 0 & 2 \end{bmatrix}$.

$Y^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. c) $[2, 4, 2]^T$ then $[4, 8, 4]^T$. We get multiplied by 2. We end up with $[32, 64, 32]^T$.

Problem 6) (10 points)

In a desperate attempt to find the **philosopher's stone**, we take a block from the periodic system of elements and row reduce it.



$$A = \begin{bmatrix} 5 & 6 & 7 & 8 & 9 & 10 \\ 13 & 14 & 15 & 16 & 17 & 18 \\ 31 & 32 & 33 & 34 & 35 & 36 \\ 49 & 50 & 51 & 52 & 53 & 54 \\ 81 & 82 & 83 & 84 & 85 & 86 \end{bmatrix} .$$

a) (2 points) As you will see, the rank of A is 2. What can you tell about the nullity of A?

b) (4 points) Find a basis for the kernel of A.

c) (4 points) Find a basis for the image of A.

Solution:
a) By the rank-nullity theorem, the nullity is 4.
b) Row reduce

$$rref(A) = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 & -4 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis of the kernel is

$$\mathcal{B}_{ker} = \{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 4 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \} .$$

c) A basis of the image is

$$\mathcal{B}_{ker} = \{ \begin{bmatrix} 5 \\ 13 \\ 31 \\ 48 \\ 18 \end{bmatrix} , \begin{bmatrix} 6 \\ 14 \\ 32 \\ 50 \\ 82 \end{bmatrix} \} .$$

Problem 7) (10 points)

a) (4 points) In order to build a **chandelier** in 4-dimensional space, we find a basis of the linear space W consisting of all vectors perpendicular to the vector

$$v = \begin{bmatrix} 2 \\ 4 \\ 5 \\ 6 \end{bmatrix} /9 \; .$$

Note that this is a vector of length 1.

b) (3 points) Find a basis of the space V of vectors perpendicular to W.

c) (3 points) Define the 4×1 matrix Q containing the unit vector v as a column vector. We know that $A = QQ^T$ is a projection matrix onto the space spanned by v. Find A.

Solution:

a) The basis of V is

$$\mathcal{B} = \{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \} \; .$$

b) The basis of V is

$$\mathcal{B} = \{ \begin{bmatrix} 2 \\ 4 \\ 5 \\ 6 \end{bmatrix} \}$$

c)

$$QQ^T = vv^T = \begin{bmatrix} \frac{4}{81} & \frac{8}{81} & \frac{10}{81} & \frac{4}{27} \\ \frac{8}{81} & \frac{16}{81} & \frac{20}{81} & \frac{8}{27} \\ \frac{10}{81} & \frac{20}{81} & \frac{25}{81} & \frac{10}{27} \\ \frac{4}{27} & \frac{8}{27} & \frac{10}{27} & \frac{4}{9} \end{bmatrix}$$

Problem 8) (10 points)

Shuri designed a protective suit for her brother **T'Challa**, king of Wakanda. The suit transforms a force vector x to a momentum vector Ax. Laboratory testing gave

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \; A \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} .$$

a) (3 points) Check that

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis of $\mathbf{R}^3$.

b) (1 point) Is $\mathcal{B}$ an orthonormal basis?

c) (3 points) The transformation in the basis $\mathcal{B}$ is given by a matrix B. Find this matrix B.

d) (3 points) Now compute the matrix A which describes the transformation in the standard basis.

Solution:

a) We have $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. This row reduces to the identity matrix.

b) No, the vectors are not perpendicular.

c)

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} .$$

d) $A = SBS^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -2 \end{bmatrix}$.

Problem 9) (10 points)

During the **big data** hype, an essay in Wired proclaimed "the end of theory", as "with enough data, the numbers speak for themselves". Others called such claims "complete bollocks". The "Financial Times" even wrote in 2014: "Big Data has arrived, but big insights have not." Anyway, whatever your take on "big data" is, there is no question that the ability to analyze data is important. In order to gain insight in the data

$$X = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \\ 2 \\ -2 \end{bmatrix}, Y = \begin{bmatrix} -3 \\ 0 \\ -3 \\ 3 \\ 0 \\ 3 \end{bmatrix}, Z = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} .$$

we compute the **variance**, $\text{Var}[X] = (X \cdot X)/6$, the **standard deviation** $\sigma[X] = \sqrt{\text{Var}[X]}$, the **covariance** $\text{Cov}[X,Y] = (X \cdot Y)/6$ and the **correlation** $\text{Cor}[X,Y] = \frac{\text{Cov}[X,Y]}{\sigma[X]\sigma[Y]}$.

a) (3 points) Find the standard deviations of X of Y and of Z.

b) (3 points) Find the covariance of X and Y, the covariance of X and Z as well as of Y and Z.

c) (4 points) Compute the correlation of X and Z as well as the correlation of Y and Z.

Solution:
a) The variances are $16/6 = 8/3$, 6 and $2/3$. The standard deviations are $\sqrt{8/3}$,$\sqrt{6}$ and $\sqrt{2/3}$.
b) The covariances are $-1, -1, 0$.
c) The correlations are $-3/4, 0$.