Name:

MWF 9 Oliver Knill
MWF 10 Jeremy Hahn
MWF 10 Hunter Spink
MWF 11 Matt Demers
MWF 11 Yu-Wen Hsu
MWF 11 Ben Knudsen
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MWF 12 Hakim Walker
TTH 10 Ana Balibanu
TTH 10 Morgan Opie
TTH 10 Rosalie Belanger-Rioux
TTH 11:30 Philip Engel
TTH 11:30 Alison Miller

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- All matrices are real matrices unless specified otherwise.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader cannot be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) True or False? No justifications are needed.

1) T F If A and B are orthogonal matrices, then so is $A+B$.

2) T F Every real 2×2-matrix has a real eigenvalue.

3) T F If $\{v_1, v_2\}$ is a basis of R^2 and A is an invertible (2 x 2)-matrix, then $\{Av_1, Av_2\}$ is also a basis of R^2.

4) T F A basis for the kernel of the operator $(D-1)^2$ on C^∞ is $\{e^{\lambda t}, te^{\lambda t}\}$ for $\lambda = 1$.

5) T F All symmetric matrices with positive matrix entries have eigenvalues which are all real and positive.

6) T F If A is a reflection matrix, then A is invertible.

7) T F If A is a symmetric $n \times n$ matrix such that $A^6 = I_n$, then A is the identity matrix I_n.

8) T F The operator $T = D^2 + 1$ defines a linear transformation $f \to Tf$ from C^∞ to C^∞.

9) T F If A and B are $n \times n$ symmetric matrices, then $A^2 - B^2$ is symmetric.

10) T F In the Fourier series expansion of the function $f(x) = x^5$ on $[-\pi, \pi]$ the coefficient a_n of $\cos(nx)$ is zero for every n.

11) T F If A and B are diagonalizable over the reals, then $A+B$ is diagonalizable over the reals.

12) T F Every 5×4 matrix has a non-zero vector in its kernel.

13) T F If A is a square matrix such that $\ker(A)$ is contained in $\mathrm{im}(A)$, then $A^2 = 0$.

14) T F The a_0 Fourier coefficient of the function $f(x) = e^{-x^2}$ is zero.

15) T F If an invertible square matrix A has two QR decompositions $A = Q_1R_1$ and $A = Q_2R_2$, then $R_1 = R_2$.

16) T F The eigenvalue 1 of a reflection at a line in $\mathbb{R}^2$ cannot have geometric multiplicity 2.

17) T F If $(d/dt)\vec{x}(t) = A\vec{x}(t)$ is an asymptotically stable continuous dynamical system, then each eigenvalue λ of A satisfies $|\lambda| < 1$.

18) T F The solutions to the heat equation $u_t(x,t) = u_{xx}(x,t)$ with $u(x,0) = \sin(14x)$ converge to 0 as $t \to \infty$.

19) T F If A is a 3×3 matrix and u, v are vectors such that $Av = 5v$ and $Au = -u$ then $A(3v + 2u) = 15v - 2u$.

20) T F There are real 2×2 matrices A and B such that $AB - BA = I_2$.

Problem 2) (10 points) No justifications needed

a) (3 points) We check some properties of the specific matrix A which is given below. Check each box which applies: (By "diagonalizable", we mean "diagonalizable over the reals"):

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Property	Check if applies
A is a projection	
A is a reflection	
A is orthogonal	
A is invertible	
A is symmetric	
A is diagonalizable	

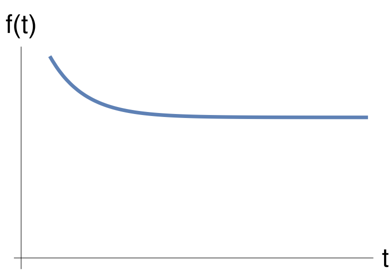
b) (3 points) Pick the statements which are true for every non-invertible 2×2 matrix A.

Statement	Check if true
The trace is zero	
The determinant is zero	
The trace of A is an eigenvalue of A	
The determinant of A is an eigenvalue	
The two columns are parallel	
The two rows are parallel	

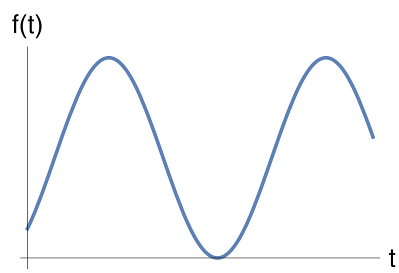
c) (4 points) Match the differential equations with possible solution graphs. Each equation matches exactly one graph.

Enter A-D	Differential equation
	$f''(t) = 1$
	$f'(t) = 1$

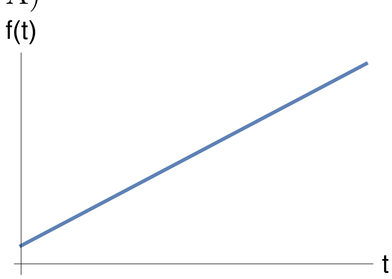
Enter A-D	Differential equation
	$f'(t) = -f(t) + 1$
	$f'''(t) = -f'(t)$



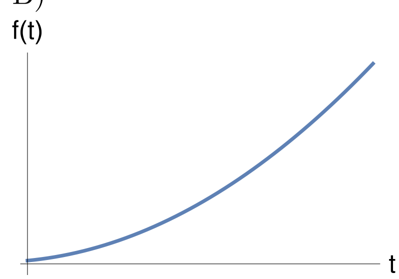
A)



B)



C)



D)

Problem 3) (10 points) No justifications needed

a) (3 points) Which of the following matrices are diagonalizable over the reals? Check each box which applies.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

b) (2 points) Which of the following two matrices has an eigenvalue which is not real? They are "still lives" in the "game of life". Which one? Enter either "Loaf" or "Boat" in the following box:

$$\text{Loaf} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Boat} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

c) (2 points) Linear or not linear? In all cases, we deal with functions f in the linear space $X = C^{\infty}_{\text{per}}([-\pi, \pi])$ for which $a_n(f), b_n(f)$ are the Fourier coefficients of f and $||f||$ is the length of the function as defined in Fourier theory.

Space	linear	nonlinear	Transformation	linear	nonlinear
$\{f \in X \mid f(1) = b_1(f)\}$			$Tf(x) = a_1(f)f$		
$\{f \in X \mid f'(1) = 0\}$			$Tf(x) = \|\|f\|\| \sin(x)$		

d) (3 points) A continuous function on $[-\pi, \pi]$ has the Fourier series

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) .$$

Check a box, if the statement above the box is true for every set of functions specified to the left.

Function	Check if $a_0 = 0$	Check if all $a_n = 0, n > 0$	Check if all $b_n = 0$
f is even			
f is odd			
f has no real roots			

Problem 4) (10 points)

We look for 4 numbers x, y, z, w. We know their sum is 20 and that their "super sum" $x-y+z-w$ is 10. As a matter of fact these two equations form a system $Ax = b$ which defines a 2-dimensional plane V in 4-dimensional space.

a) (6 points) Find the solution space of all these numbers by row reducing its augmented matrix $B = [A|b]$ carefully.

$$B = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 20 \\ 1 & -1 & 1 & -1 & 10 \end{array}\right] .$$

b) (4 points) Find two linearly independent vectors which are perpendicular to the kernel of A.

Problem 5) (10 points)

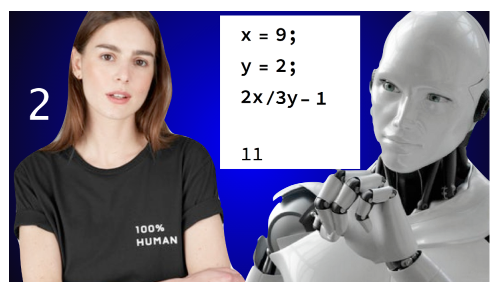
People on social media have been in war about expressions like $\boxed{2x/3y - 1 \text{ if } x = 9 \text{ and } y = 2}$. Computers and humans disagree: most humans get 2, while most machines return 11. A psychologist investigates whether the size of the numbers influences the answer and asks people. (Previous research has shown that machines are not impressed by size). This needs data fitting:

Using the least square method, find those a and b such that

$$\frac{ax}{3y} - b = 2$$

best fits the data points in the following table:

x	y
9	3
6	1
-3	1
0	1



Problem 6) (10 points)

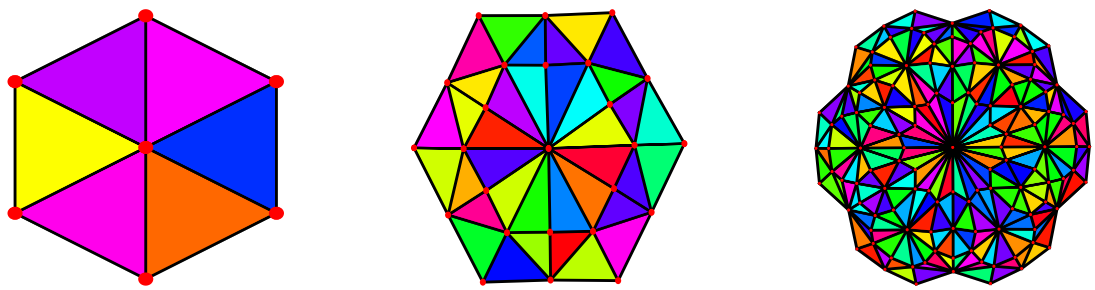
Let $\vec{x} = \begin{bmatrix} v \\ e \\ f \end{bmatrix}$ denote the number of vertices, edges and faces of a polyhedron. During a

Barycentric refinement, this vector transforms as

$$A\vec{x} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix} \vec{x} \; .$$

a) (5 points) Verify that $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ are eigenvectors of A and find their eigenvalues.

b) (5 points) Write down a closed form solution of the discrete dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with the initial condition $\begin{bmatrix} v \\ e \\ f \end{bmatrix} = \begin{bmatrix} 7 \\ 12 \\ 6 \end{bmatrix}$.

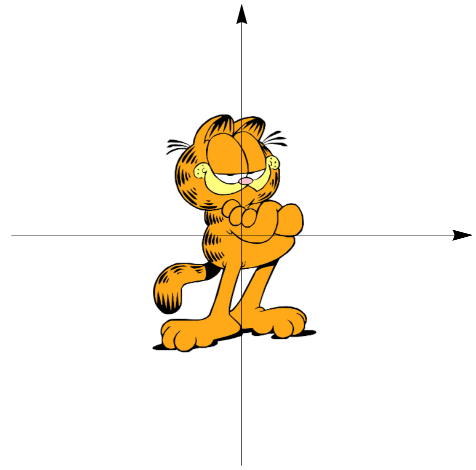


Problem 7) (10 points)

The **Arnold cat map** is $T\vec{v} = A\vec{v}$ where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} .$$

It is an icon of chaos theory.



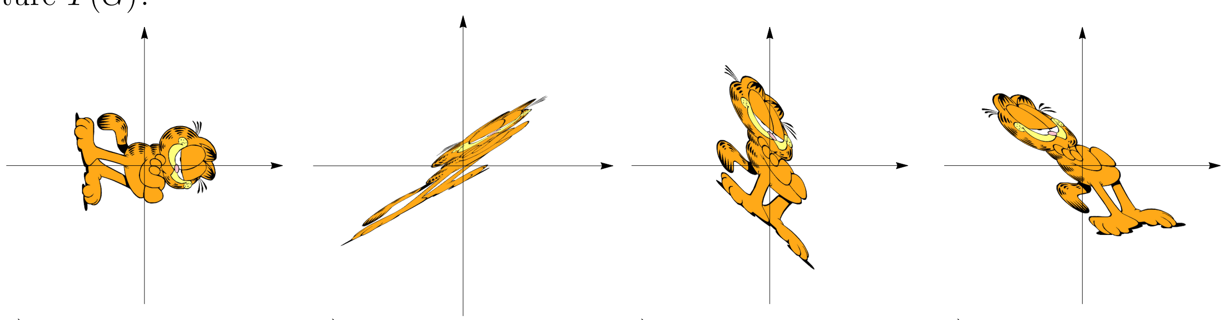
a) (2 points) What is the characteristic polynomial of A?

b) (2 points) Find the eigenvalues and eigenvectors of A. Don't expect integer answers.

c) (2 points) Is the discrete dynamical system defined by A asymptotically stable or not?

d) (2 points) Write down an orthogonal matrix S and a diagonal matrix B such that $B = S^{-1}AS$.

e) (2 points) Garfield G seen to the right has been transformed by the cat map T. Which one of the pictures a),b),c),d) represents the transformed picture $T(G)$?



a) b) c) d)

Problem 8) (10 points)

The following configuration is called the "Beacon Oscillator" in the **Game of Life**.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) (2 points) What is the rank and the nullity of A?

b) (4 points) Find a basis for the kernel and a basis for the image of A.

c) (4 points) The following matrix is called the "glider configuration" in the **Game of life**.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} .$$

Find the inverse of A using row reduction.

Problem 9) (10 points)

Remember to give computation details. Answers alone can not be given credit.

a) (2 points) The following matrix displays the solution of the Cellular automaton 10. Find its determinant

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

b) (2 points) Find the determinant of

$$B = \begin{bmatrix} 0 & 0 & 3 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

c) (2 points) Find the determinant of

$$C = \begin{bmatrix} 1 & 2 & 3 & 8 & 8 \\ 4 & 5 & 0 & 8 & 8 \\ 6 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} .$$

d) (2 points) Find the determinant of

$$D = \begin{bmatrix} 11 & 2 & 3 & 2 & 1 \\ 1 & 12 & 3 & 2 & 1 \\ 1 & 2 & 13 & 2 & 1 \\ 1 & 2 & 3 & 12 & 1 \\ 1 & 2 & 3 & 2 & 11 \end{bmatrix} .$$

e) (2 points) Find the determinant of $E = 2Q + 5Q^{-1} + 7I$: (you can leave it in terms of eigenvalues of the basic circulant matrix Q you have seen. No simplifications are required):

$$E = \begin{bmatrix} 7 & 2 & 0 & 0 & 5 \\ 5 & 7 & 2 & 0 & 0 \\ 0 & 5 & 7 & 2 & 0 \\ 0 & 0 & 5 & 7 & 2 \\ 2 & 0 & 0 & 5 & 7 \end{bmatrix} .$$

Problem 10) (10 points)

Find the general solution to the following differential equations:

a) (1 point)

$$f'(t) = 1/(t+1)$$

b) (1 point)

$$f''(t) = e^t + t$$

c) (2 points)

$$f''(t) + f(t) = t + 2$$

d) (2 points)

$$f''(t) - 2f'(t) + f(t) = e^t$$

e) (2 points)

$$f''(t) - f(t) = e^t + \sin(t)$$

f) (2 points)

$$f''(t) - f(t) = e^{-3t}$$

Problem 11) (10 points)

We consider the nonlinear system of differential equations

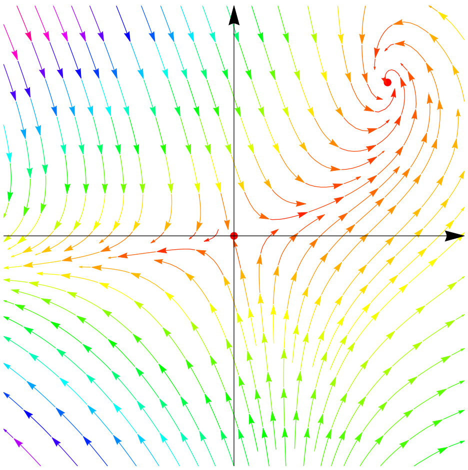
$$\begin{aligned} \frac{d}{dt}x &= x + y - xy \\ \frac{d}{dt}y &= x - 3y + xy . \end{aligned}$$

a) (2 points) Find the equilibrium points.

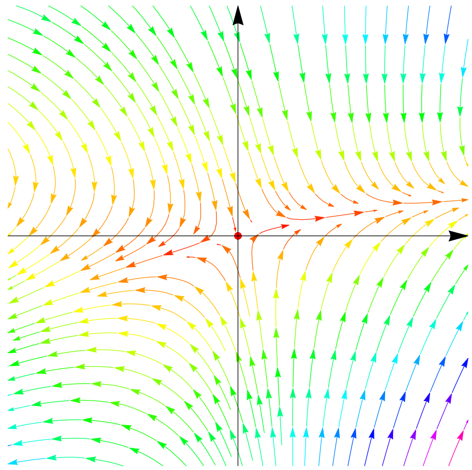
b) (3 points) Find the Jacobian matrix at each equilibrium point.

c) (3 points) Use the Jacobean matrix at an equilibrium to determine for each equilibrium point whether it is stable or not.

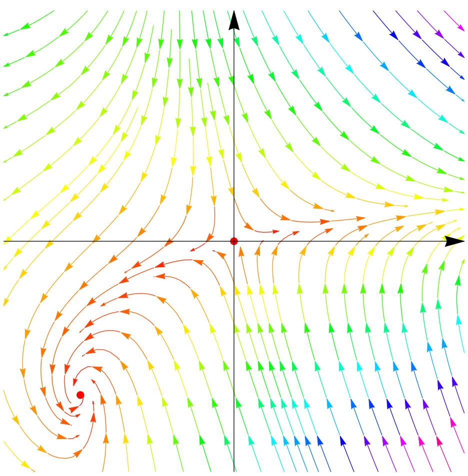
d) (2 points) Which of the diagrams A-D is the phase portrait of the system above?



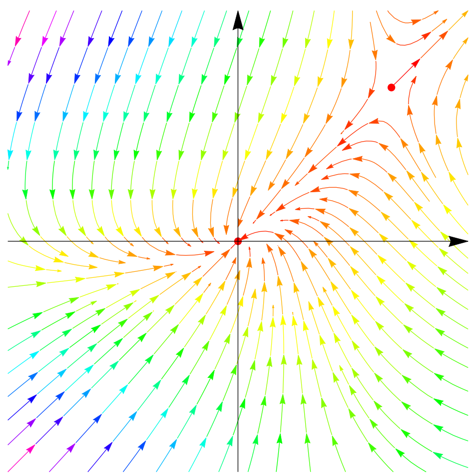
A



B



C



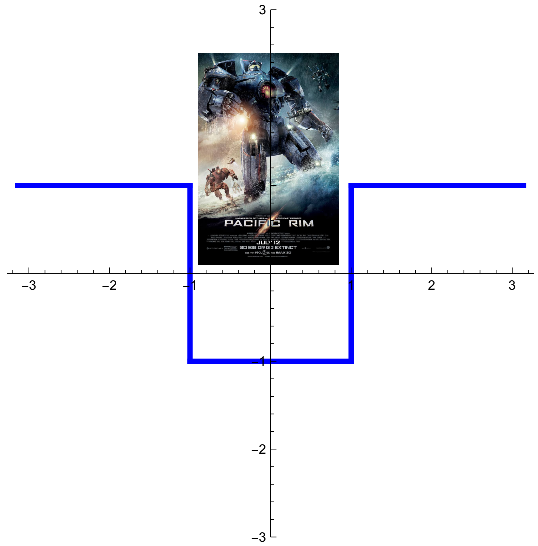
D

Problem 12) (10 points)

a) (6 points) Find the **Fourier series** of the function which is 1 if $|x| > 1$ and -1 else. We call it the **Pacific rim** function.

$$f(x) = \begin{cases} 1 & |x| > 1 \\ -1 & |x| \leq 1 \end{cases} .$$

The graph of the function f on $[-\pi, \pi]$ is displayed to the right.



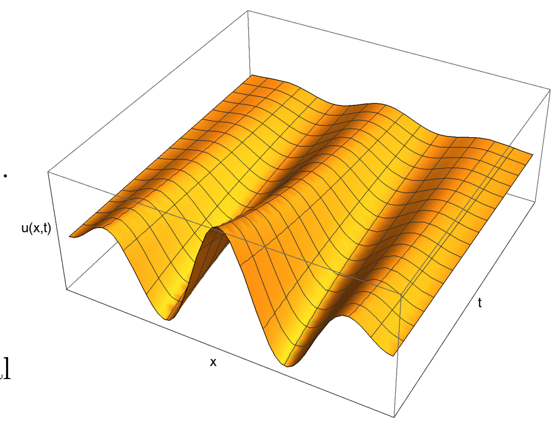
b) (4 points) Find the value of the sum of the squares of all the Fourier coefficients of f.

Problem 13) (10 points)

a) (3 points) Solve the standard **heat equation**

$$u_t = 9u_{xx}$$

with initial condition $u(x, 0) = 4\sin(x) + 5\sin(2x)$.



b) (2 points) Use Parseval's equality to find $||u(x, 1)||^2$.

c) (5 points) Solve the **modified wave equation**

$$u_{tt} = 9u_{xx} - 2u + 1$$

with initial condition $u(x, 0) = 7\sin(5x)$ and the initial velocity

$$u_t(x, 0) = x .$$