Final Exam, May 9, 2017 **Math 21b, Spring 2017**

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) (20 points) True or False? No justifications are needed.

1) [T] [F] All symmetric matrices with positive entries are diagonalizable.

2) [T] [F] If A is a symmetric matrix, then A is invertible.

3) [T] [F] If A is a symmetric matrix such that $A^5 = 0$, then $A = 0$.

4) [T] [F] If A and B are $n \times n$ symmetric matrices, then $A + B$ is symmetric.

5) [T] [F] If A and B are $n \times n$ symmetric matrices, then AB is symmetric.

6) [T] [F] If A is 2×2 matrix with $\det(A) < 0$, then the system $\frac{dx}{dt} = Ax$ has $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as a stable equilibrium.

7) [T] [F] If A is any matrix, then both AA^T and A^TA are diagonalizable.

8) [T] [F] The matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.

9) [T] [F] Every 2×3 matrix has a non-zero vector in its kernel.

10) [T] [F] If the image of A is contained in the kernel of a square matrix A, then $A^2 = 0$.

11) [T] [F] If a 2×2 matrix A is diagonalizable $A^7 + A + I_2$ is diagonalizable.

12) [T] [F] The function $f(x) = 7\sin(5x)$ has the Fourier coefficient $b_7 = 5$ if $f(x) = a_0/\sqrt{2} + \sum_n a_n \cos(nx) + b_n \sin(nx)$.

13) [T] [F] The space of smooth functions f satisfying $f(x) = x + \sin(f(x))$ forms a linear space.

14) [T] [F] If the geometric multiplicity of the eigenvalue 1 is 2 for a 2×2 matrix A then A is the identity matrix.

15) [T] [F] Let A, B be arbitrary 2×2 matrices. Then some eigenvalue of AB is the product of some eigenvalues of A and B.

16) [T] [F] If a continuous dynamical system $x'(t) = Ax(t)$ with a 2×2 symmetric matrix A is asymptotically stable then $\det(A) > 0$.

17) [T] [F] If $\vec{x}(t+1) = A\vec{x}(t)$ is an asymptotically stable dynamical system, then each eigenvalue λ of A satisfies $|\lambda| < 1$.

18) [T] [F] The function $f(x,t) = \sin(5x)e^{-25t}$ solves the heat equation $f_t = f_{xx}$.

19) [T] [F] The length $||6\sin(175x) + 8\cos(199x)||$ is equal to 10.

20) [T] [F] All solutions to the differential equation $x''(t) + x(t) = \sin(t)$ stay bounded for $t \to \infty$.

Problem 2) (10 points) No justifications needed

a) (4 points) Which properties do hold for the following matrix? (Remember the circular matrices?)

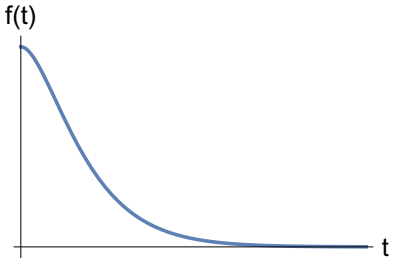
$$A = \begin{bmatrix} 100 & 2 & 3 & 4 \\ 4 & 100 & 2 & 3 \\ 3 & 4 & 100 & 2 \\ 2 & 3 & 4 & 100 \end{bmatrix}$$

Property	Check if applies
The matrix A is symmetric.	
The matrix A has eigenvector $[1, i, i^2, i^3]^T$.	
The matrix A has an eigenvalue $100 + 2i - 3 - 4i$.	
The matrix A is invertible.	
The matrix A is diagonalizable over the complex numbers.	
The matrix A is diagonalizable over the real numbers.	

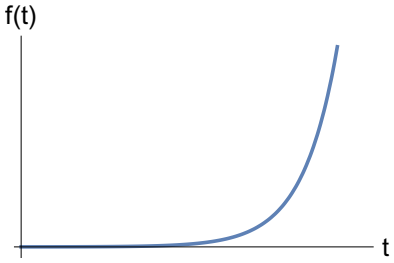
b) (3 points) Match the differential equations with possible solution graphs. There is an exact match.

Enter A-D	Differential equation
	$f'(t) + f(t) = e^{-t}$
	$f'(t) - f(t) = e^{-t}$

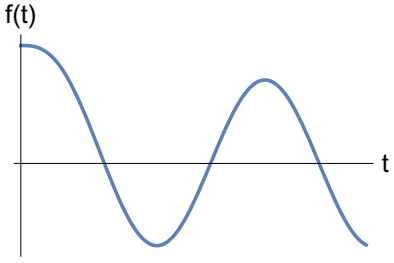
Enter A-D	Differential equation
	$f''(t) + f(t) = e^{-t}$
	$f''(t) = e^{-t}$



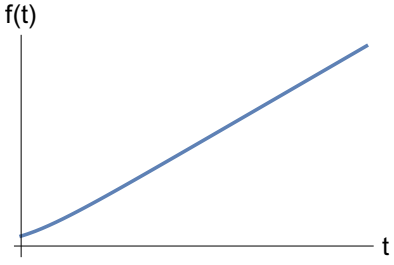
A)



B)



C)



D)

c) (3 points) Pick the statements which are true for a 3×3 matrix A.

Statement	Check if true
If A is diagonalizable and A has real eigenvalues then A is symmetric	
If A is symmetric, then A is diagonalizable and A has real eigenvalues	
If A has only eigenvalues of algebraic multiplicity 1, then A is diagonalizable	
If A is diagonalizable, then all eigenvalues of A have algebraic multiplicity 1	

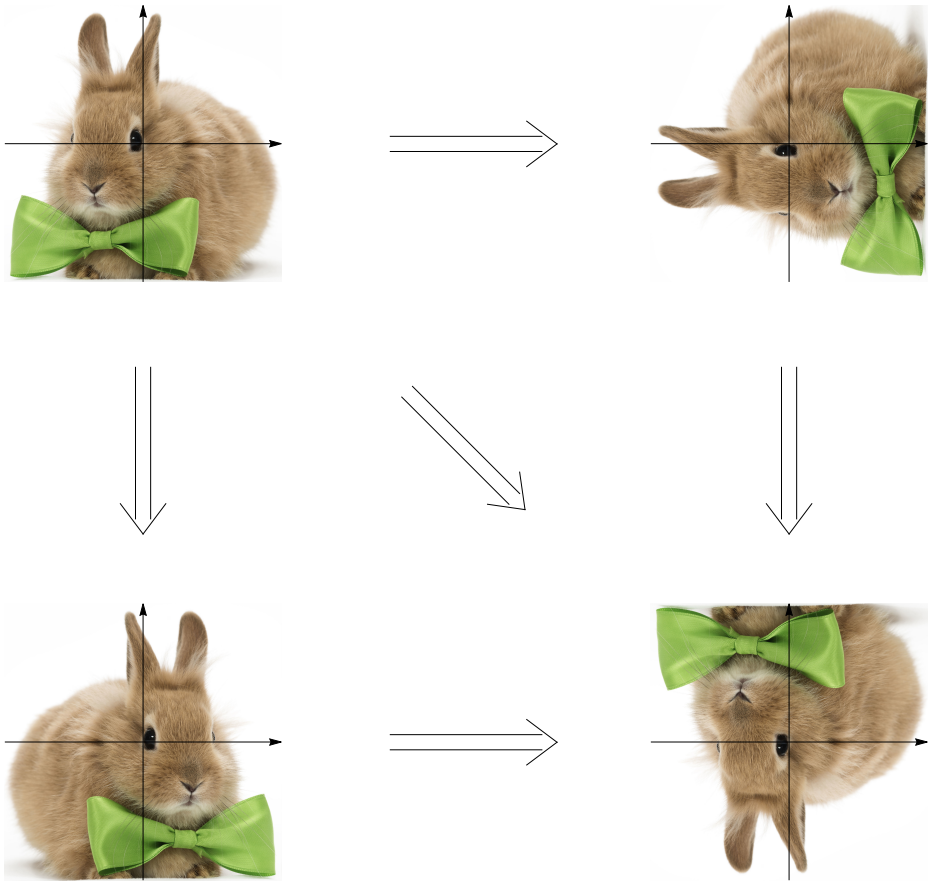
Problem 3) (10 points) No justifications needed

a) (2 points) Linear or not linear?

Space	linear	nonlinear
$\{f \in C^\infty \mid f(1) = 1\}$		
$\{f \in C^\infty \mid f'(1) = 0\}$		

Transformation	linear	nonlinear
$Tf(x) = f'(x)\sin(x)$		
$Tf(x) = f'(3)f(x)$		

b) (5 points) Match each of the 5 arrows with a matrix that does the indicated transformation. Place the name of the matrix $A-F$ near each arrow. Of course, one of the matrices is not used.



A=$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ B=$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ C=$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ D=$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ E=$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ F=$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

c) (3 points) Label each function with the most accurate description of its Fourier series

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \ .$$

Function	Check if $a_0 = 0$	Check if all $a_n = 0$	Check if all $b_n = 0$
$f(x) = \sin^3(x)$			
$f(x) = \sin^2(x)$			
$f(x) = \sin^2(x) + \sin^3(x)$			

Problem 4) (10 points)

a) (8 points) Solve the following linear system $A\vec{x} = \vec{b}$ of equations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

by row reducing the augmented matrix. We want you to perform the actual row reduction steps.

b) (2 points) Change one single entry in the first row of the matrix A leaving the vector $\vec{b}$ alone to get a matrix B so that the modified system $B\vec{x} = \vec{b}$ has no solution.

The **binary Pascal matrix** A was constructed row by row, recursively adding up all the entries up left or up but taking remainders when dividing by 2.

The picture shows **Blaise Pascal** (1623 - 1662), who in 1653 produced a tabular representation of Binomial coefficients called today the **Pascal triangle**.

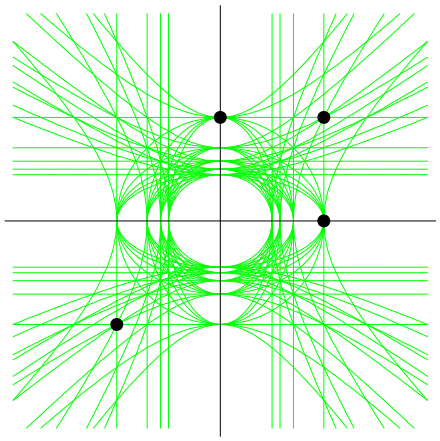
Problem 5) (10 points)

Using the least square method, find the function

$$ax^2 + by^2 = 1$$

which best fits the data points in the following table

x	y
1	0
0	1
1	1
-1	-1



Problem 6) (10 points)

You have an **investment account** x and a **credit account** y and at time $t = 0$ the account data: $x(0) = 2000$, $y(0) = 0$. The time dynamics of the portfolio is described by the following matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} .$$

We look at two models, the case of discrete time and the case of continuous time.

a) (4 points) Find a closed form solution to the problem

$$\begin{bmatrix} x(t+1) \\ y(t+1) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

with the initial condition given above.

b) (4 points) Find a closed form solution to the problem

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

with the same given initial condition.

c) (2 points) In which of the two portfolios does $x(t)$ grow faster? Is it a) or b)? No explanations are needed in c).

Problem 7) (10 points)

The following magic square is called **Chautisa Yantra**:

$$A = \begin{bmatrix} 7 & 12 & 1 & 14 \\ 2 & 13 & 8 & 11 \\ 16 & 3 & 10 & 5 \\ 9 & 6 & 15 & 4 \end{bmatrix} .$$

It has been inscribed into a stone during the 10th century and can now be found in the Parshvanath temple in Khajuraho, India.

a) (3 points) The vector $\begin{bmatrix} -1 \\ -3 \\ 1 \\ 3 \end{bmatrix}$ is an eigenvector of A. What is the corresponding eigenvalue?

b) (3 points) The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What is its corresponding eigenvalue?

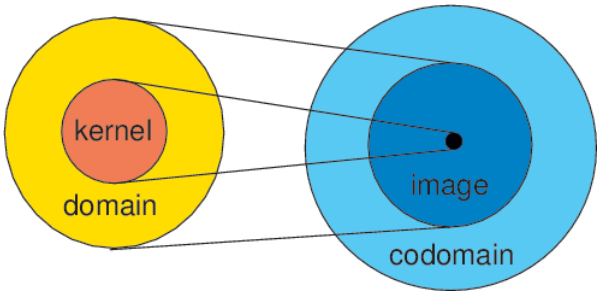
c) (2 points) What is the sum of all the four eigenvalues $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ of A?

d) (2 points) What is the determinant of A?

Problem 8) (10 points)

a) (4 points) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$



b) (3 points) Find a basis for the kernel of A.

c) (3 points) Find a basis for the image of A.

Problem 9) (10 points)

To celebrate the launch of our new determinant **bumper sticker**, we solve some determinants!
As always, we need not only your answer to the problem but also the path which led to the solution.

a) (2 points) Find the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

b) (2 points) Find the determinant of

$$B = \begin{bmatrix} 9 & 2 & 2 & 2 \\ 2 & 9 & 2 & 2 \\ 2 & 2 & 9 & 2 \\ 2 & 2 & 2 & 9 \end{bmatrix}.$$

c) (2 points) Find the determinant of

$$C = \begin{bmatrix} 2 & 8 & 5 & 2 \\ 2 & 9 & 2 & 2 \\ 2 & 8 & 9 & 2 \\ 2 & 2 & 1 & 2 \end{bmatrix}.$$

d) (2 points) Find the determinant of

$$D = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 2 & 5 & 4 \\ 2 & 5 & 4 & 3 \end{bmatrix}.$$

e) (2 points) Find the determinant of

$$E = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 2 & 3 & 2 & 3 \\ 4 & 1 & 4 & 1 \end{bmatrix}.$$

Problem 10) (10 points)

You are a chef in the first season of the new reality TV series "21b's kitchen". Solve the following differential equations:

a) (2 points)

$$f''(t) = t$$

b) (3 points)

$$f''(t) + f(t) = t$$

c) (3 points)

$$f''(t) - f(t) = t$$

d) (2 points) Check for each of the following two equations whether it has only solutions which remain bounded for $t \to \infty$? We need no explanation in d).

Differential equation	has only bounded solutions for $t \to +\infty$
$f''(t) + f'(t) = \cos(t)$	
$f''(t) - f'(t) = \cos(t)$	

Problem 11) (10 points)

We consider the nonlinear system of differential equations

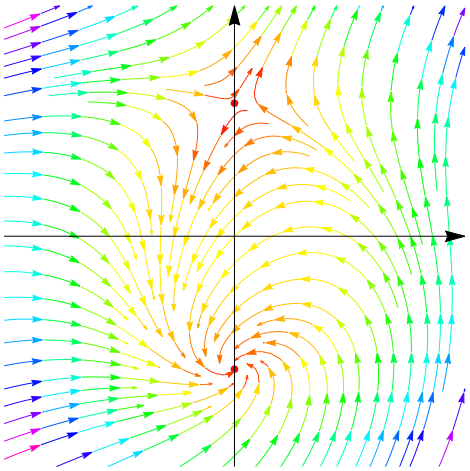
$$\begin{aligned} \frac{d}{dt}x &= (x-1)^2 + y^2 - 10 \\ \frac{d}{dt}y &= (x+1)^2 + y^2 - 10 \, . \end{aligned}$$

a) (2 points) Find the equilibrium points.

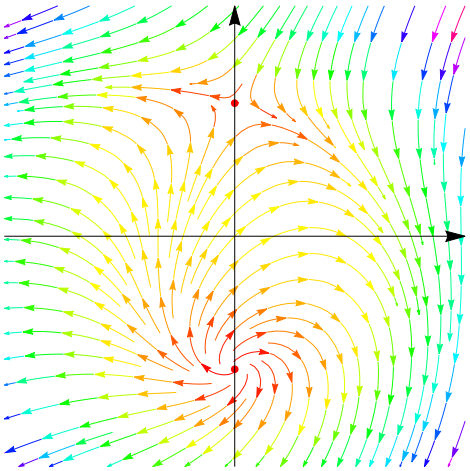
b) (3 points) Find the Jacobian matrix at each equilibrium point.

c) (3 points) Analyze the stability at each equilibrium point.

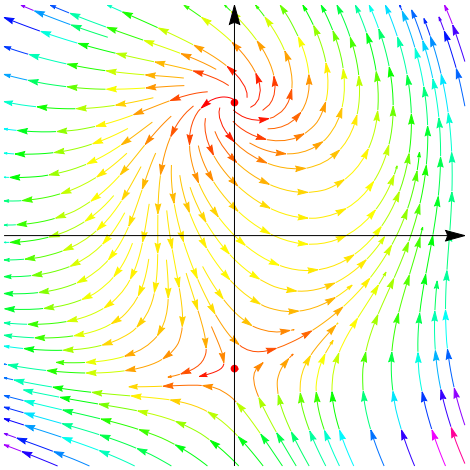
d) (2 points) Which of the following four phase portraits A-D matches?



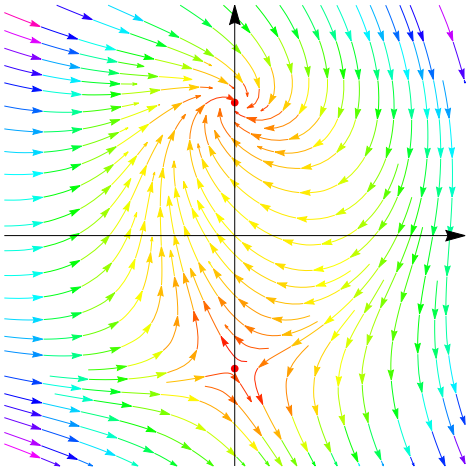
A



B



C



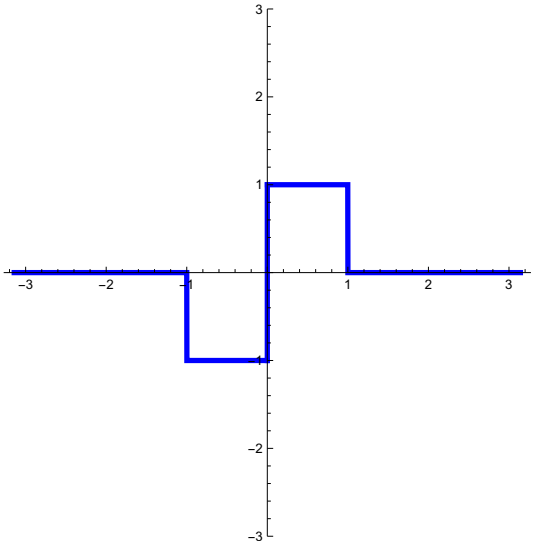
D

Problem 12) (10 points)

a) (6 points) Find the **Fourier series** of the function

$$f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ -1 & -1 \leq x \leq 0 \\ 0 & |x| > 1 \end{cases} .$$

The graph of the function f on $[-\pi, \pi]$ is displayed to the right.



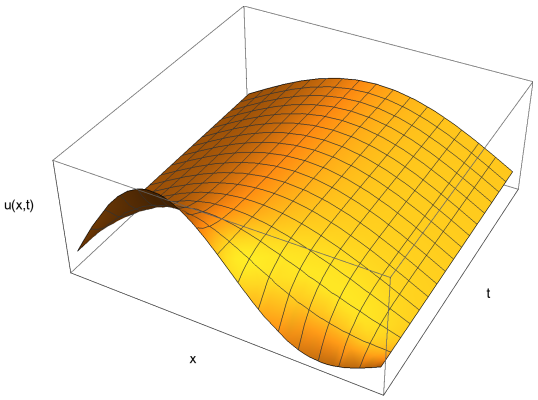
b) (4 points) Find the value of the sum of the squares of all the Fourier coefficients of f.

Problem 13) (10 points)

a) (5 points) Solve the **modified heat equation**

$$u_t = u_{xx} - u_{xxxx} + t$$

with initial condition $u(x, 0) = 4\sin(x) + 5\sin(2x)$. As usual, first find a solution of the PDE $u_t = u_{xx} - u_{xxxx}$ then add a particular solution.



b) (5 points) Solve the **modified wave equation**

$$u_{tt} = u_{xx} - u_{xxxx} + t$$

with initial condition $u(x, 0) = 7\sin(5x)$ and the initial velocity

$$u_t(x, 0) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx) .$$