

The set of piecewise smooth functions $f(x)$ on $[-\pi, \pi]$ form a linear space X . There is an **inner product** in X defined by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$$

It allows to define angles, length, distance, projections in X as we did in finite dimensions.

THE FOURIER BASIS.

THEOREM. The functions $\{\cos(nx), \sin(nx), 1/\sqrt{2}\}$ form an orthonormal set in X .

Proof. You check the details in the homework. To check linear independence a few integrals need to be computed. For all $n, m \geq 1$, with $n \neq m$ you have to show:

$$\begin{aligned} \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle &= 1 \\ \langle \cos(nx), \cos(nx) \rangle &= 1, \langle \cos(nx), \cos(mx) \rangle = 0 \\ \langle \sin(nx), \sin(nx) \rangle &= 1, \langle \sin(nx), \sin(mx) \rangle = 0 \\ \langle \sin(nx), \cos(mx) \rangle &= 0 \\ \langle \sin(nx), 1/\sqrt{2} \rangle &= 0 \\ \langle \cos(nx), 1/\sqrt{2} \rangle &= 0 \end{aligned}$$

To verify the above integrals, the following trigonometric identities are useful:

$$\begin{aligned} 2 \cos(nx) \cos(my) &= \cos(nx - my) + \cos(nx + my) \\ 2 \sin(nx) \sin(my) &= \cos(nx - my) - \cos(nx + my) \\ 2 \sin(nx) \cos(my) &= \sin(nx + my) + \sin(nx - my) \end{aligned}$$

FOURIER COEFFICIENTS. The **Fourier coefficients** of a function f in X are defined as

$$a_0 = \langle f, 1/\sqrt{2} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)/\sqrt{2} dx$$

$$a_n = \langle f, \cos(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \langle f, \sin(nt) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

FOURIER SERIES. The **Fourier representation** of a smooth function f is the identity

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

We take it for granted that the series converges and that the identity holds at all points x where f is continuous.

ODD AND EVEN FUNCTIONS. The following advice can save you time when computing Fourier series:

If f is odd: $f(x) = -f(-x)$, then f has a sin series.

If f is even: $f(x) = f(-x)$, then f has a cos series.

If you integrate an odd function over $[-\pi, \pi]$ you get 0.

The product of two odd functions is even, the product between an even and an odd function is odd.

EXAMPLE 1. Let $f(x) = x$ on $[-\pi, \pi]$. This is an odd function ($f(-x) + f(x) = 0$) so that it has a sin series: with $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{-1}{\pi} (x \cos(nx)/n + \sin(nx)/n^2)|_{-\pi}^{\pi} = 2(-1)^{n+1}/n$, we get $x = \sum_{n=1}^{\infty} 2 \frac{(-1)^{n+1}}{n} \sin(nx)$. If we evaluate both sides at a point x , we obtain identities. For $x = \pi/2$ for example, we get

$$\frac{\pi}{2} = 2\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots\right)$$

is a **formula of Leibnitz**.

EXAMPLE 2. Let $f(x) = \cos(x) + 1/7 \cos(5x)$. This **trigonometric polynomial** is already the Fourier series. There is no need to compute the integrals. The nonzero coefficients are $a_1 = 1, a_5 = 1/7$.

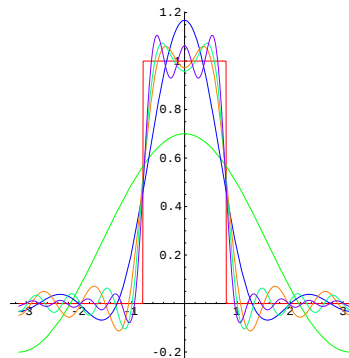
EXAMPLE 3. Let $f(x) = 1$ on $[-\pi/2, \pi/2]$ and $f(x) = 0$ else. This is an even function $f(-x) = f(x)$ so that it has a cos series: with $a_0 = 1/(\sqrt{2}), a_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 1 \cos(nx) dx = \frac{\sin(nx)}{\pi n} \Big|_{-\pi/2}^{\pi/2} = \frac{2(-1)^n}{\pi(2m+1)}$ if $n = 2m + 1$ is odd and 0 else. So, the series is

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \left(\frac{\cos(x)}{1} - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - \dots \right)$$

Remark. The function in Example 3 is not smooth, but Fourier theory still works. What happens at the discontinuity point $\pi/2$? The Fourier series converges to $1/2$. Diplomatically it has chosen the point in the middle of the limits from the right and the limit from the left.

FOURIER APPROXIMATION. For a smooth function f , the Fourier series of f converges to f . The Fourier coefficients are the coordinates of f in the Fourier basis.

The function $f_n(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^n a_k \cos(kx) + \sum_{k=1}^n b_k \sin(kx)$ is called a **Fourier approximation** of f . The picture to the right plots a few approximations in the case of a piecewise continuous even function given in example 3).



THE PARSEVAL EQUALITY. When evaluating the square of the length of f with the square of the length of the series, we get

$$\|f\|^2 = a_0^2 + \sum_{k=1}^{\infty} a_k^2 + b_k^2.$$

EXAMPLE. We have seen in example 1 that $f(x) = x = 2(\sin(x) - \sin(2x)/2 + \sin(3x)/3 - \sin(4x)/4 + \dots$. Because the Fourier coefficients are $b_k = 2(-1)^{k+1}/k$, we have $4(1 + 1/4 + 1/9 + \dots) = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = 2\pi^2/3$ and so

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = \frac{\pi^2}{6}$$

Isn't it fantastic that we can sum up the reciprocal squares? This formula has been obtained already by **Leonard Euler**. The problem was called the **Basel problem**.

FUNCTIONS OF TWO VARIABLES. We consider functions $f(x, t)$ which are for fixed t a piecewise smooth function in x . Analogously as we studied the motion of a **vector** $\vec{v}(t)$, we are now interested in the motion of a **function** f in time t . While the governing equation for a vector was an ordinary differential equation $\dot{x} = Ax$ (ODE), the describing equation is now be a **partial differential equation** (PDE) $\dot{f} = T(f)$. The function $f(x, t)$ could denote the **temperature of a stick** at a position x at time t or the **displacement of a string** at the position x at time t . The motion of these dynamical systems will be easy to describe in the orthonormal Fourier basis $1/\sqrt{2}, \sin(nx), \cos(nx)$ treated in an earlier lecture.

PARTIAL DERIVATIVES. We write $f_x(x, t)$ and $f_t(x, t)$ for the **partial derivatives** with respect to x or t . The notation $f_{xx}(x, t)$ means that we differentiate twice with respect to x .

Example: for $f(x, t) = \cos(x + 4t^2)$, we have

- $f_x(x, t) = -\sin(x + 4t^2)$
- $f_t(x, t) = -8t \sin(x + 4t^2)$.
- $f_{xx}(x, t) = -\cos(x + 4t^2)$.

One also uses the notation $\frac{\partial f(x, y)}{\partial x}$ for the partial derivative with respect to x . Tired of all the "partial derivative signs", we always write $f_x(x, t)$ for the partial derivative with respect to x and $f_t(x, t)$ for the partial derivative with respect to t .

PARTIAL DIFFERENTIAL EQUATIONS. A partial differential equation is an equation for an unknown function $f(x, t)$ in which different partial derivatives occur.

- $f_t(x, t) + f_x(x, t) = 0$ with $f(x, 0) = \sin(x)$ has a solution $f(x, t) = \sin(x - t)$.
- $f_{tt}(x, t) - f_{xx}(x, t) = 0$ with $f(x, 0) = \sin(x)$ and $f_t(x, 0) = 0$ has a solution $f(x, t) = (\sin(x - t) + \sin(x + t))/2$.

THE HEAT EQUATION. The temperature distribution $f(x, t)$ in a metal bar $[0, \pi]$ satisfies the **heat equation**

$$f_t(x, t) = \mu f_{xx}(x, t)$$

This partial differential equation tells that the rate of change of the temperature at x is proportional to the second space derivative of $f(x, t)$ at x . The function $f(x, t)$ is assumed to be zero at both ends of the bar and $f(x) = f(x, 0)$ is a given initial temperature distribution. The constant μ depends on the heat conductivity properties of the material. Metals for example conduct heat well and would lead to a large μ .

REWRITING THE PROBLEM. We can write the problem as

$$\frac{d}{dt}f = \mu D^2 f$$

We will solve the problem in the same way as we solved linear differential equations:

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

where A is a matrix - by diagonalization.

We use that the Fourier basis is just the diagonalization: $D^2 \cos(nx) = (-n^2) \cos(nx)$ and $D^2 \sin(nx) = (-n^2) \sin(nx)$ show that $\cos(nx)$ and $\sin(nx)$ are eigenfunctions to D^2 with eigenvalue $-n^2$. By a symmetry trick, we can focus on sin-series from now on.

SOLVING THE HEAT EQUATION WITH FOURIER THEORY. The heat equation $f_t(x, t) = \mu f_{xx}(x, t)$ with smooth $f(x, 0) = f(x)$, $f(0, t) = f(\pi, t) = 0$ has the solution

$$f(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 \mu t}$$

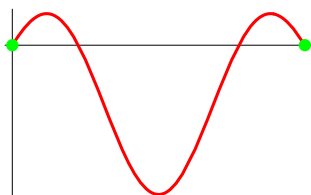
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

Proof: With the initial condition $f(x) = \sin(nx)$, we have the evolution $f(x, t) = e^{-\mu n^2 t} \sin(nx)$. If $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ then $f(x, t) = \sum_{n=1}^{\infty} b_n e^{-\mu n^2 t} \sin(nx)$.

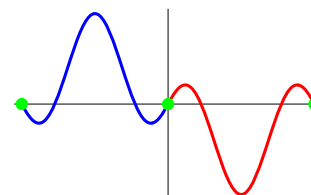
A SYMMETRY TRICK. Given a function f on the interval $[0, \pi]$ which is zero at 0 and π . It can be extended to an odd function on the doubled interval $[-\pi, \pi]$.

The Fourier series of an odd function is a pure sin-series. The Fourier coefficients are $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$.

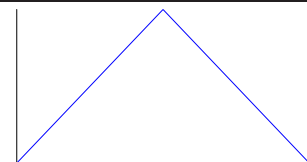
The function is given on $[0, \pi]$.



The odd symmetric extension on $[-\pi, \pi]$.



EXAMPLE. Assume the initial temperature distribution $f(x, 0)$ is a sawtooth function which has slope 1 on the interval $[0, \pi/2]$ and slope -1 on the interval $[\pi/2, \pi]$. We first compute the sin-Fourier coefficients of this function.

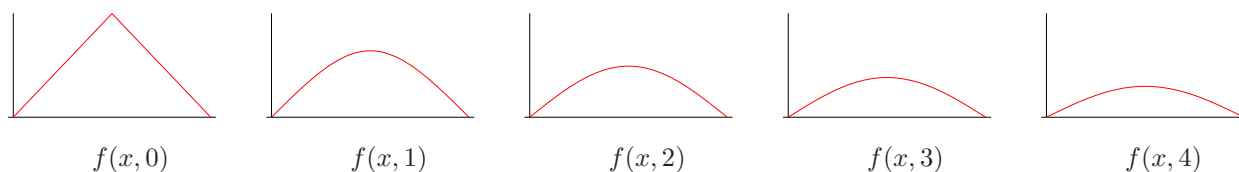


The sin-Fourier coefficients are $b_n = \frac{4}{n^2 \pi} (-1)^{(n-1)/2}$ for odd n and 0 for even n . The solution is

$$f(x, t) = \sum_n b_n e^{-\mu n^2 t} \sin(nx) .$$

The exponential term containing the time makes the function $f(x, t)$ converge to 0: The body cools. The higher frequencies are damped faster: "smaller disturbances are smoothed out faster."

VISUALIZATION. We can plot the graph of the function $f(x, t)$ or slice this graph and plot the temperature distribution for different values of the time t .



THE WAVE EQUATION. The height of a string $f(x, t)$ at time t and position x on $[0, \pi]$ satisfies the **wave equation**

$$f_{tt}(x, t) = c^2 f_{xx}(x, t)$$

where c is a constant. As we will see, c is the **speed** of the waves.

REWRITING THE PROBLEM. We can write the problem as

$$\frac{d^2}{dt^2} f = c^2 D^2 f$$

We will solve the problem in the same way as we solved

$$\frac{d^2}{dt^2} \vec{x} = A \vec{x}$$

If A is diagonal, then every basis vector x satisfies an equation of the form $\frac{d^2}{dt^2} x = -c^2 x$ which has the solution $x(t) = x(0) \cos(ct) + x'(0) \sin(ct)/c$.

SOLVING THE WAVE EQUATION WITH FOURIER THEORY. The wave equation $f_{tt} = c^2 f_{xx}$ with $f(x, 0) = f(x), f_t(x, 0) = g(x), f(0, t) = f(\pi, t) = 0$ has the solution

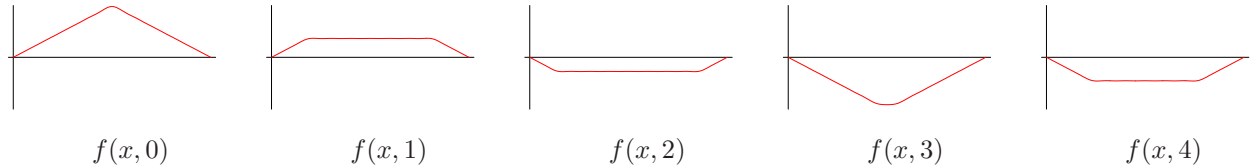
$$f(x, t) = \sum_{n=1}^{\infty} a_n \sin(nx) \cos(nct) + \frac{b_n}{nc} \sin(nx) \sin(nct)$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

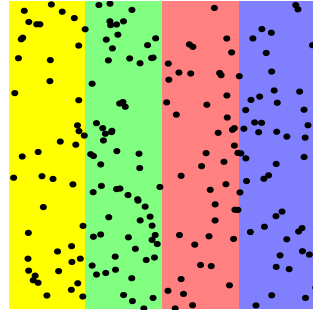
$$b_n = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(nx) dx$$

Proof: With $f(x) = \sin(nx), g(x) = 0$, the solution is $f(x, t) = \cos(nct) \sin(nx)$. With $f(x) = 0, g(x) = \sin(nx)$, the solution is $f(x, t) = \frac{1}{nc} \sin(nct) \sin(nx)$. For $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ and $g(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, we get the formula by summing these two solutions.

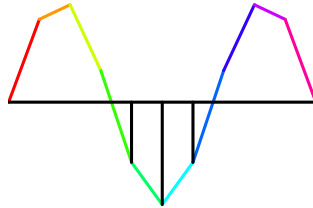
VISUALIZATION. We can just plot the graph of the function $f(x, t)$ or plot the string for different times t .



TO THE DERIVATION OF THE HEAT EQUATION. The temperature $f(x, t)$ is proportional to the kinetic energy at x . Divide the stick into n adjacent cells and assume that in each time step, a fraction of the particles moves randomly either to the right or to the left. If $f_i(t)$ is the **energy** of particles in cell i at time t , then the energy $f_i(t + dt)$ of particles at time $t + dt$ is proportional to $f_i(t) + (f_{i-1}(t) - 2f_i(t) + f_{i+1}(t))$. Therefore the **discrete time derivative** $f_i(t + dt) - f_i(t) \sim dt f_t$ is equal to the **discrete second space derivative** $dx^2 f_{xx}(x, t) \sim (f(x + dx, t) - 2f(x, t) + f(x - dx, t))$.



TO THE DERIVATION OF THE WAVE EQUATION. We can model a string by n discrete particles linked by springs. Assume that the particles can move up and down only. If $f_i(t)$ is the **height** of the particles, then the right particle pulls with a force $f_{i+1} - f_i$, the left particle with a force $f_{i-1} - f_i$. Again, $(f_{i-1}(t) - 2f_i(t) + f_{i+1}(t))$ which is a discrete version of the second derivative because $f_{xx}(x, t) dx^2 \sim (f(x + dx, t) - 2f(x, t) + f(x - dx, t))$.



OVERVIEW: The heat and wave equation can be solved like ordinary differential equations:

Ordinary differential equations

$$x_t(t) = Ax(t)$$

$$x_{tt}(t) = Ax(t)$$

Diagonalizing A leads for eigenvectors $\vec{v}$

$$Av = -c^2v$$

to the differential equations

$$v_t = -c^2v$$

$$v_{tt} = -c^2v$$

which are solved by

$$v(t) = e^{-c^2t}v(0)$$

$$v(t) = v(0)\cos(ct) + v_t(0)\sin(ct)/c$$

Partial differential equations

$$f_t(x, t) = f_{xx}(x, t)$$

$$f_{tt}(x, t) = f_{xx}(x, t)$$

Diagonalizing $T = D^2$ with eigenfunctions $f(x) = \sin(nx)$

$$Tf = -n^2f$$

leads to the differential equations

$$f_t(x, t) = -n^2f(x, t)$$

$$f_{tt}(x, t) = -n^2f(x, t)$$

which are solved by

$$f(x, t) = f(x, 0)e^{-n^2t}$$

$$f(x, t) = f(x, 0)\cos(nt) + f_t(x, 0)\sin(nt)/n$$

NOTATION:

f function on $[-\pi, \pi]$ smooth or piecewise smooth.

t time variable

x space variable

D the differential operator $Df(x) = f'(x) = d/dxf(x)$.

T linear transformation, like $Tf = D^2f = f''$.

c speed of the wave.

$Tf = \lambda f$ Eigenvalue equation analog to $Av = \lambda v$.

f_t partial derivative of $f(x, t)$ with respect to time t .

f_x partial derivative of $f(x, t)$ with respect to space x .

f_{xx} second partial derivative of f twice with respect to space x .

μ heat conductivity

$f(x) = -f(-x)$ odd function, has sin Fourier series