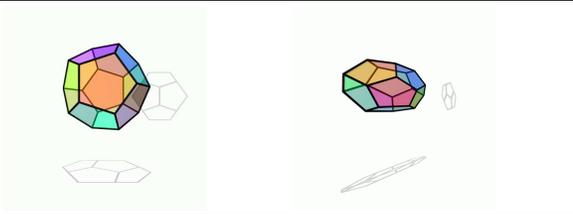


10/6/2003, LINEAR TRAFOS IN GEOMETRY

Math 21b, O. Knill

HW: section 2.2: 6,8,16,30,34,47\*,50\*

LINEAR TRANSFORMATIONS DEFORMING A BODY



A CHARACTERIZATION OF LINEAR TRANSFORMATIONS: a transformation  $T$  from  $\mathbf{R}^n$  to  $\mathbf{R}^m$  which satisfies  $T(\vec{0}) = \vec{0}$ ,  $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$  and  $T(\lambda\vec{x}) = \lambda T(\vec{x})$  is a linear transformation.

**Proof.** Call  $\vec{v}_i = T(\vec{e}_i)$  and define  $S(\vec{x}) = A\vec{x}$ . Then  $S(\vec{e}_i) = T(\vec{e}_i)$ . With  $\vec{x} = x_1\vec{e}_1 + \dots + x_n\vec{e}_n$ , we have  $T(\vec{x}) = T(x_1\vec{e}_1 + \dots + x_n\vec{e}_n) = x_1T(\vec{e}_1) + \dots + x_nT(\vec{e}_n) = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$  as well as  $S(\vec{x}) = A(x_1\vec{e}_1 + \dots + x_n\vec{e}_n) = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$  proving  $T(\vec{x}) = S(\vec{x}) = A\vec{x}$ .

SHEAR:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{[Diagram: square to parallelogram]} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{[Diagram: square to parallelogram]}$$

In general, shears are transformation in the plane with the property that there is a vector  $\vec{w}$  such that  $T(\vec{w}) = \vec{w}$  and  $T(\vec{x}) - \vec{x}$  is a multiple of  $\vec{w}$  for all  $\vec{x}$ . If  $\vec{u}$  is orthogonal to  $\vec{w}$ , then  $T(\vec{x}) = \vec{x} + (\vec{u} \cdot \vec{x})\vec{w}$ .

SCALING:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{[Diagram: square to larger square]} \quad A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \text{[Diagram: square to smaller square]}$$

One can also look at transformations which scale  $x$  differently then  $y$  and where  $A$  is a diagonal matrix.

REFLECTION:

$$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix} \quad \text{[Diagram: reflection across a line]} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{[Diagram: reflection across the x-axis]}$$

Any reflection at a line has the form of the matrix to the left. A reflection at a line containing a unit vector  $\vec{u}$  is  $T(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$  with matrix  $A = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$

PROJECTION:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{[Diagram: projection onto the x-axis]} \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{[Diagram: projection onto the y-axis]}$$

A projection onto a line containing unit vector  $\vec{u}$  is  $T(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$  with matrix  $A = \begin{bmatrix} u_1u_1 & u_2u_1 \\ u_1u_2 & u_2u_2 \end{bmatrix}$

ROTATION:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{[Diagram: reflection through origin]} \quad A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \quad \text{[Diagram: rotation counter-clockwise]}$$

Any rotation has the form of the matrix to the right.

ROTATION-DILATION:

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \quad \text{[Diagram: rotation and dilation]} \quad A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{[Diagram: general rotation-dilation matrix]}$$

A rotation dilation is a composition of a rotation by angle  $\arctan(y/x)$  and a dilation by a factor  $\sqrt{x^2 + y^2}$ . If  $z = x + iy$  and  $w = a + ib$  and  $T(x, y) = (X, Y)$ , then  $X + iY = zw$ . So a rotation dilation is tied to the process of the multiplication with a complex number.

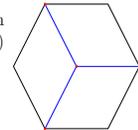
BOOST:

$$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix} \quad \text{[Diagram: Lorentz boost]} \quad \text{The boost is a basic Lorentz transformation in special relativity. It acts on vectors } (x, ct), \text{ where } t \text{ is time, } c \text{ is the speed of light and } x \text{ is space.}$$

Unlike in Galileo transformation  $(x, t) \mapsto (x + vt, t)$  (which is a shear), time  $t$  also changes during the transformation. The transformation has the effect that it changes length (Lorentz contraction). The angle  $\alpha$  is related to  $v$  by  $\tanh(\alpha) = v/c$ . One can write also  $A(x, ct) = ((x + vt)/\gamma, t + (v/c^2)\gamma x)$ , with  $\gamma = \sqrt{1 - v^2/c^2}$ .

ROTATION IN SPACE. Rotations in space are defined by an axes of rotation and an angle. A rotation by  $120^\circ$  around a line containing  $(0, 0, 0)$  and  $(1, 1, 1)$

belongs to  $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$  which permutes  $\vec{e}_1 \rightarrow \vec{e}_2 \rightarrow \vec{e}_3$ .



REFLECTION AT PLANE. To a reflection at the  $xy$ -plane belongs the matrix

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$  as can be seen by looking at the images of  $\vec{e}_i$ . The picture to the right shows the textbook and reflections of it at two different mirrors.



PROJECTION ONTO SPACE. To project a 4d-object into xyz-space, use

for example the matrix  $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ . The picture shows the projection of the four dimensional cube (tesseract, hypercube) with 16 edges  $(\pm 1, \pm 1, \pm 1, \pm 1)$ . The tesseract is the theme of the horror movie "hypercube".

