

FUNCTION SPACES II

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Homework: Section 4.1: 4,14,42,50,52,38*,56*

LINEAR SPACES. The three main examples of linear spaces are linear subspaces of \mathbf{R}^n , spaces of matrices and spaces of functions.

- The n -dimensional space R^n .
- linear subspaces of R^n like the trivial space $\{0\}$, lines or planes.
- M_n , the space of all square $n \times n$ matrices.
- P_n , the space of all polynomials of degree n .
- The space P of all polynomials.
- C^∞ , the space of all smooth functions on the line
- C^0 , the space of all continuous functions on the line.

BASIS.

A set \mathcal{B} of elements in a linear space X is called a **basis**, if the vectors span the space and if they are linearly independent.

EXAMPLE 1.

The set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis of the linear space M_2 of all 2×2 matrices.

Check. We can write any matrix as a linear combination of these 4 matrices. If a linear combination of these matrices is zero, then it must be the trivial linear combination.

EXAMPLE 2.

The set

$$\mathcal{B} = \{ \cos(x), \sin(x) \}$$

is a basis for the linear space

$$X = \{ f \in C^\infty \mid f'' = -f \}$$

EXAMPLE 3.

The set

$$\mathcal{B} = \{ 1, x, x^2, x^3 \}$$

is a basis for all polynomials of degree ≤ 3 .

EXAMPLE 4.

Find a basis for the space of all polynomials of degree ≤ 5 which are even $f(x) = f(-x)$. It contains for example $f(x) = x^2$ but the space does not contain x^3 for example.

PROBLEM.

Find a basis for the linear space of all polynomials for which $f(3) = 0$ and $f(10) = 0$ and $f''' = 0$.

$f''' = 0$ means that the space is a subspace of $P_2 = \{ ax^2 + bx + c \}$. We have

$$a3^2 + b3 + c = 0, a10^3 + b10 + c = 0$$

which is a system of linear equations. We have a free variable c and get $a = 7c/2910, b = -991c/2910$ which shows that X has the basis

$$\mathcal{B} = \{ 7x^2 - 991x + 1 \}.$$

PROBLEM.

Find the kernel of the derivative D on the space of polynomials of degree 3.

PROBLEM.

Find the kernel of the trace transformation on M_2 which you have seen in a practice exam.

EXAMPLE.

Is the set of matrices belonging to horizontal shears a linear space?

EXAMPLE.

Is the set of dilation matrices a linear space? What is its dimension?

EXAMPLE.

Is the set of rotation-dilation matrices

$$X = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right\}$$

a linear space? What is its dimension?

EXAMPLE.

Is the set of reflection-dilation matrices

$$X = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \right\}$$

a linear space? What is its dimension?

DIFFERENTIAL EQUATIONS.

The space

$$X = \{ af'' + bf' + cf = 0 \}$$

is the **solution space** of the **second order homogeneous differential equation**. We are interested in function spaces because we want to understand such solution spaces. We will later see that the solution space is two dimensional. You can assume this for the following two problems.

EXAMPLE.

Find a basis for the solution space $\{ f \in C^\infty \mid f'' = f \}$.

EXAMPLE.

Find a basis for the solution space $\{ f \in C^\infty \mid f'' - 7f' + 12f = 0 \}$.