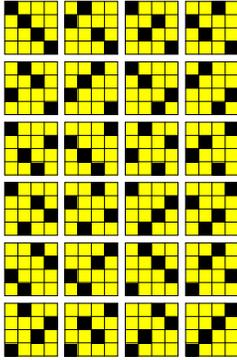Section Section 6.1: 8,18,40,44,Problem A,42*,56*

PERMUTATIONS. A **permutation** of $\{1, 2, \ldots, n\}$ is a rearrangement of $\{1, 2, \ldots, n\}$. There are $n! = n \cdot (n-1) \ldots \cdot 1$ different permutations of $\{1, 2, \ldots, n\}$: fixing the position of first element leaves $(n-1)!$ possibilities to permute the rest.

EXAMPLE. There are 6 permutations of $\{1, 2, 3\}$: $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$.

PATTERNS AND SIGN. The matrix A with zeros everywhere except $A_{i,\pi(i)} = 1$ is called a permutation matrix or the **pattern** of π. An **inversion** is a pair $k < l$ such that $\pi(k) < \pi(l)$. The **sign** of a permutation π is defined as $(-1)^{\text{inv}(\pi)}$ where $\text{inv}(\pi)$ is the number inversions in the pattern of π. To get the sign in a permutation count the number $\text{inv}(\pi)$ of pairs of black squares, where the upper square is to the right.
EXAMPLES. $\text{inv}(1, 2) = 0$, $\text{inv}(2, 1) = 1$. $\text{inv}(1, 2, 3) = \text{inv}(3, 2, 1) = \text{inv}(2, 3, 1) = 1$. $\text{inv}(1, 3, 2) = \text{inv}(3, 2, 1) = \text{inv}(2, 1, 3) = -1$.

DETERMINANT The **determinant** of a $n \times n$ matrix A is defined as the sum
$\boxed{\sum_\pi (-1)^{\text{inv}(\pi)} A_{1\pi(1)} A_{2\pi(2)} \cdots A_{n\pi(n)}}$, where π is a permutation of $\{1, 2, \ldots, n\}$.

2×2 CASE. The determinant of $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$. There are two permutations of $(1, 2)$. The identity permutation $(1, 2)$ gives $A_{11}A_{12}$, the permutation $(2, 1)$ gives $A_{21}A_{22}$.
If you have seen some multi-variable calculus, you know that $\det(A)$ is the area of the parallelogram spanned by the column vectors of A. The two vectors form a basis if and only if $\det(A) \neq 0$.

3×3 CASE. The determinant of $A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is $aei + bfg + cdh - ceg - fha - bdi$ corresponding to the 6 permutations of $(1, 2, 3)$. Geometrically, $\det(A)$ is the volume of the parallelepiped spanned by the column vectors of A. The three vectors form a basis if and only if $\det(A) \neq 0$.

EXAMPLE DIAGONAL AND TRIANGULAR MATRICES. The determinant of a diagonal or triangular matrix is the product of the diagonal elements.

EXAMPLE PERMUTATION MATRICES. The determinant of a matrix which has everywhere zeros except $A_{i\pi(j)} = 1$ is just the sign $(-1)^{\text{inv}(\pi)}$ of the permutation.

HOW FAST CAN WE COMPUTE THE DETERMINANT?.
The cost to find the determinant is the same as for the Gauss-Jordan elimination as we will see below The graph to the left shows some measurements of the time needed for a CAS to calculate the determinant in dependence on the size of the $n \times n$ matrix. The matrix size ranges from n=1 to n=300. We also see a best cubic fit of these data using the least square method from the last lesson. It is the cubic $p(x) = a + bx + cx^2 + dx^3$ which fits best through the 300 data points.

WHY DO WE CARE ABOUT DETERMINANTS?

- check invertibility of matrices
- have geometric interpretation as volume
- explicit algebraic expressions for inverting a matrix
- as a natural functional on matrices it appears in formulas in particle or statistical physics

- allow to define orientation in any dimensions
- appear in change of variable formulas in higher dimensional integration.
- proposed alternative concepts are unnatural, hard to teach and and harder to understand
- determinants are fun

TRIANGULAR AND DIAGONAL MATRICES.
The determinant of a **diagonal** or **triangular** matrix is the product of its diagonal elements.

Example: $\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix} \right) = 20$.

PARTITIONED MATRICES.
The determinant of a **partitioned matrix** $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the product $\det(A)\det(B)$.

Example $\det\left(\begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \right) = 2 \cdot 12 = 24$.

LINEARITY OF THE DETERMINANT. If the columns of A and B are the same except for the i'th column,
$$\boxed{\det([v_1, \ldots, v, \ldots v_n]) + \det([v_1, \ldots, w, \ldots v_n]) = \det([v_1, \ldots, v + w, \ldots v_n])}$$

In general, one has $\boxed{\det([v_1, \ldots, kv, \ldots v_n]) = k \det([v_1, \ldots, v, \ldots v_n])}$ The same identities hold for rows and follow directly from the original definition of the determinant.

PROPERTIES OF DETERMINANTS.
$\boxed{\det(AB) = \det(A)\det(B)}$ $\boxed{\det(SAS^{-1}) = \det(A)}$ $\boxed{\det(\lambda A) = \lambda^n \det(A)}$
$\boxed{\det(A^{-1}) = \det(A)^{-1}}$ $\boxed{\det(A^T) = \det(A)}$ $\boxed{\det(-A) = (-1)^n \det(A)}$
If B is obtained from A by switching two rows, then $\det(B) = -\det(A)$. If B is obtained by adding an other row to a given row, then this does not change the value of the determinant.

PROOF OF $\det(AB) = \det(A)\det(B)$, one brings the $n \times n$ matrix $[A|AB]$ into row reduced echelon form. Similar than the augmented matrix $[A|b]$ was brought into the form $[1|A^{-1}b]$, we end up with $[1|A^{-1}AB] = [1|B]$. By looking at the $n \times n$ matrix to the left during Gauss-Jordan elimination, the determinant has changed by a factor $\det(A)$. We end up with a matrix B which has determinant $\det(B)$. Therefore, $\det(AB) = \det(A)\det(B)$.
PROOF OF $\det(A^T) = \det(A)$. The transpose of a pattern is a pattern with the same signature.

PROBLEM. Find the determinant of $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \end{bmatrix}$.   SOLUTION. Three row transpositions give $B = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ a matrix which has determinant 84. Therefore $\det(A) = (-1)^3 \det(B) = -84$.

PROBLEM. Determine $\det(A^{100})$, where A is the matrix $\begin{vmatrix} 1 & 2 \\ 3 & 16 \end{vmatrix}$.

SOLUTION. $\det(A) = 10$, $\det(A^{100}) = (\det(A))^{100} = 10^{100} = 1 \cdot$ gogool. This name as well as the gogoolplex $= 10^{10^{100}}$ are official. They are huge numbers: the mass of the universe for example is $10^{52} kg$ and $1/10^{10^{51}}$ is the chance to find yourself on Mars by quantum fluctuations. (R.E. Crandall, Scient. Amer., Feb. 1997).

ROW REDUCED ECHELON FORM. Determining rref(A) also determines $\det(A)$.
If A is a matrix and α_i are the factors which are used to scale different rows and s is the number of times, two rows are switched, then $\boxed{\det(A) = (-1)^s \alpha_1 \cdots \alpha_n \det(\text{rref}(A))}$.

INVERTIBILITY. Because of the last formula: $\boxed{\text{A } n \times n \text{ matrix } A \text{ is invertible if and only if } \det(A) \neq 0.}$

THE LAPLACE EXPANSION. (This is the **definition of determinants** of the book.) We compute the determinant of $n \times n$ matrices $A = a_{ij}$. Choose a column i. For each entry a_{ji} in that column, take the $(n-1) \times (n-1)$ matrix A_{ij} called **minor** which does not contain the i'th column and j'th row. One gets
$$\boxed{\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \cdots + (-1)^{i+n} a_{in} \det(A_{in}) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})}$$

This Laplace expansion just arranges the permutations: listing all permutations of the form $(1, *, \ldots, *)$ of n elements is the same then listing all permutations of $(2, *, \ldots, *)$ of $(n-1)$ elements.

ORTHOGONAL MATRICES. Because $Q^T Q = 1$, we have $\det(Q)^2 = 1$ and so $|\det(Q)| = 1$ Rotations have determinant 1, reflections can have determinant -1.

QR DECOMPOSITION. If $A = QR$, then $\det(A) = \det(Q)\det(R)$. The determinant of Q is ± 1, the determinant of R is the product of the diagonal elements of R.