LINEAR MAPS ON FUNCTION SPACES — Math 21b, O. Knill

Homework: 4.2: 28,40,34,58,66,78*

FUNCTION SPACES REMINDER. A linear space X has the property that if f, g are in X, then $f + g, \lambda f$ and a zero vector" 0 are in X.

- P_n, the space of all polynomials of degree n.
- The space P of all polynomials.
- $C^\infty(R)$, the space of all smooth functions on the line
- $C^\infty_{per}(R)$ the space of all 2π periodic functions.

In all these function spaces, the function $f(x) = 0$ which is constant 0 is the zero function.

LINEAR TRANSFORMATIONS. A map T on a linear space X is called **a linear transformation** if the following three properties hold $T(x + y) = T(x) + T(y), T(\lambda x) = \lambda T(x)$ and $T(0) = 0$. Examples are:

- $Df(x) = f'(x)$ on C^∞
- $Tf(x) = \int_0^x f(x)\, dx$ on C^∞.
- $Tf(x) = f(2x)$.
- $Tf(x) = \sin(x) f(x)$ on C^∞
- $Tf(x) = 5f(x)$
- $Tf(x) = f(x - 1)$.

SUBSPACES, EIGENVALUES, BASIS, KERNEL, IMAGE are defined as before

X linear subspace	$f, g \in X, f + g \in X, \lambda f \in X, 0 \in X$.
T linear transformation	$T(f + g) = T(f) + T(g), T(\lambda f) = \lambda T(f), T(0) = 0$.
$f_1, f_2, ..., f_n$ linear independent	$\sum_i c_i f_i = 0$ implies $f_i = 0$.
$f_1, f_2, ..., f_n$ span X	Every f is of the form $\sum_i c_i f_i$.
$f_1, f_2, ..., f_n$ basis of X	linear independent and span.
T has eigenvalue λ	$Tf = \lambda f$
kernel of T	$\{Tf = 0\}$
image of T	$\{Tf \mid f \in X\}$.

Some concepts do not work without modification. Example: $\det(T)$ or $\mathrm{tr}(T)$ are not always defined for linear transformations in infinite dimensions. The concept of a basis in infinite dimensions also needs to be defined properly.

DIFFERENTIAL OPERATORS. The differential operator D which takes the derivative of a function f can be iterated: $D^n f = f^{(n)}$ is the n'th derivative. A linear map $T(f) = a_n f^{(n)} + ... + a_1 f + a_0$ is called a differential operator. We will next time study linear systems

$$Tf = g$$

which are the analog of systems $A\vec{x} = \vec{b}$. Differential equations of the form $Tf = g$, where T is a differential operator is called a higher order differential equation.

EXAMPLE: INTEGRATION. Solve

$$Df = g .$$

The linear transformation T has a one dimensional kernel, the linear space of constant functions. The system $Df = g$ has therefore infinitely many solutions. Indeed, the solutions are of the form $f = G + c$, where F is the anti-derivative of g.

EXAMPLE: FIND THE IMAGE AND KERNEL OF D. Look at $X = C^\infty(R)$. The kernel consists of all functions which satisfy $f'(x) = 0$. These are the constant functions. The kernel is one dimensional. The image is the entire space X because we can solve $Df = g$ by integration.

EXAMPLE: Find the eigenvectors to the eigenvalue λ of the operator D on $C^\infty(R)$. We have to solve

$$Df = \lambda f .$$

We see that $f(x) = e^\lambda(x)$ is a solution. The operator D has every real or complex number λ as an eigenvalue.

EXAMPLE: Find the eigenvectors to the eigenvalue λ of the operator D on $C^\infty(T)$. We have to solve

$$Df = \lambda f .$$

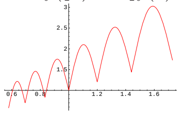
We see that $f(x) = e^\lambda(x)$ is a solution. But it is only a periodic solution if $\lambda = 2k\pi i$. Every number $\lambda = 2\pi k i$ is an eigenvalue. Eigenvalues are "quantized".

EXAMPLE: THE HARMONIC OSCILLATOR. When we solved the harmonic oscillator differential equation

$$D^2 f + f = 0 .$$

last week, we actually saw that the transformation $T = D^2 + 1$ has a two dimensional kernel. It is spanned by the functions $f_1(x) = \cos(x)$ and $f_2(x) = \sin(x)$. Every solution to the differential equation is of the form $c_1 \cos(x) + c_2 \sin(x)$.

AN ODE FROM QUANTUM CALCULUS. Define the q-derivative $D_q f(x) = d_q f(x)/d_q(x)$, where $d_q(f)(x) = f(qx) - f(x)$, where $q > 1$ is close to 1. To solve the quantum differential equation $D_q f = f$, we have to find the kernel of $T(f)(x) = f(qx) - f(x) - (q - 1)f(x)$. which simplifies to $f(qx) = qf(x)$. A differentiation gives $f'(qx) = f'(x)$ which has the linear functions $f(x) = ax$ as solutions. More functions can be obtaind by taking an arbitrary function $g(t)$ on the interval $[1, q)$ satisfying $f(q) = qf(1)$ and extending it to the other intervals $[q^k, q^{k+1}]$ using the rule $f(q^k x) = q^k f(x)$. All these solutions grow linearly.



EXAMPLE: EIGENVALUES OF $T(f) = f(x + \alpha)$ on $C^\infty(T)$, where α is a real number. This is not easy to find but one can try with functions $f(x) = e^{inx}$. Because $f(x + \alpha) = e^{in(x+\alpha)} = e^{inx} e^{in\alpha}$. we see that $e^{in\alpha} = \cos(n\alpha) + i \sin(n\alpha)$ are indeed eigenvalues. If α is irrational, there are infinitely many.

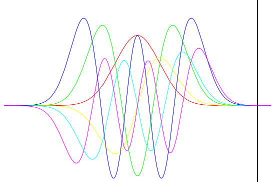
EXAMPLE: THE QUANTUM HARMONIC OSCILLATOR. We have to find the eigenvalues and eigenvectors of

$$T(f) = D^2 f - x^2 f - 1$$

The function $f(x) = e^{-x^2/2}$ is an eigenfunction to the eigenvalue 0. It is called the **vacuum**. Physicists know a trick to find more eigenvalues: write $P = D$ and $Qf = xf$. Then $Tf = (P - Q)(P + Q)f$. Because $(P + Q)(P - Q)f = Tf + 2f = 2f$ we get by applying $(P - Q)$ on both sides

$$(P - Q)(P + Q)(P - Q)f = 2(P - Q)f$$

which shows that $(P - Q)f$ is an eigenfunction to the eigenvalue 2. We can repeat this construction to see that $(P - Q)^n f$ is an eigenfunction to the eigenvalue $2n$.



BOURNE SUPREMACY. One can compute with differential operators as with matrices. What is $e^{Dt} f$? If we expand, we see $e^{Dt} f = f + Dtf + D^2 t^2 f/2! + D^3 t^3 f/3! + ...$ Because the differential equation $d/dtf = Df = d/dxf$ has the solution $f(t, x) = f(x + t)$ as well as $e^{Dt} f$, we have proven the **Taylor theorem**

$$f(x + t) = f(x) + tf'(x)/1! + tf''(x)/2! + ...$$

This is the ultimate **supreme** way to prove that theorem (one still has to worry about the convergence of the right hand side). By the way, in quantum mechanics iD is the momentum operator. In quantum mechanics, an operator H generates the motion $e^{iH} f$. The Taylor theorem tells us that the momentum operator is the generator for translation in quantum mechanics. If that does not give an "identity" to that dreadful theorem in calculus!