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- Start by writing your name in the above box and check your section in the box to the left.

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. (The actual exam will have more free space). If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or un-staple the packet.

- Please write neatly. Answers which are illegible for the grader can not be given credit.

- No notes, books, calculators, computers, or other electronic aids can be allowed.

- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) (20 points) True or False? No justifications are needed.

T F	If a linear system has 2 equations and 3 unknowns, then we either have infinitely many or no solutions.
T F	If S is an invertiable matrix which contains the vectors $\vec{v}_1, ..., \vec{v}_n$ as columns, then $\vec{v}_1, ..., \vec{v}_n$ is a basis of $\mathbf{R}^n$.
T F	If A, B are given $n \times n$ matrices, then the formula $(A-B)(A+B) = A^2 - B^2$ holds.
T F	Suppose A is an $m \times n$ matrix, where $n < m$. If the rank of A is m, then there is a vector $y \in \mathbf{R}^m$ for which the system $Ax = y$ has no solutions.
T F	The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 3 \end{bmatrix}$ is invertible.
T F	The rank of an lower-triangular matrix equals the number of non-zero entries along the diagonal.
T F	The row reduced echelon form of a 3×3 matrix of rank 2 is one of the following $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.
T F	The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a shear.
T F	For any matrix A, one has $\dim(\ker(A)) = \dim(\ker(\text{rref}(A)))$.
T F	If $\ker(A)$ is included in $\text{im}(A)$, then A is not invertible.
T F	There exists an invertible 3×3 matrix, for which 7 of the 9 entries are π.
T F	The dimension of the image of a matrix A is equal to the dimension of the image of the matrix $\text{rref}(A)$.
T F	There exists an invertible $n \times n$ matrix whose inverse has rank $n - 1$.
T F	If A and B are $n \times n$ matrices, then AB is invertible if and only if both A and B are invertible.
T F	There exist matrices A, B such that A has rank 4 and B has rank 7 and AB has rank 5.
T F	There exist matrices A, B such that A has rank 2 and B has rank 7 and AB has rank 1.
T F	If for an invertible matrix A one has $A^2 = A$, then $A = I_2$.
T F	If an invertible matrix A satisfies $A^2 = 1$, then $A = I_2$ or $A = -I_2$.
T F	The matrix $\begin{bmatrix} c-1 & -1 \\ 2 & c+1 \end{bmatrix}$ is invertible for every real number c.
T F	For 2×2 matrices A and B, if $AB = 0$, then either $A = 0$ or $B = 0$.
T F	The plane $x + y - z = 1$ is a linear subspace of three dimensional space.
T F	If T is a rotation in space with an angle $\pi/6$ around the z axes, then the linear transformation $S(x) = T(x) - x$ is invertible.

Problem 2) (10 points)

Match each of matrices with one of the geometric descriptions below. You don't have to give explanations.

Matrix	Enter A-H here.
a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
b) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	
d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	

Matrix	Enter A-H here.
e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	
f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
g) $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	
h) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	

A) Shear along a plane.
B) Projection onto a plane.
C) Rotation around an axes.
D) Reflection at a point.
E) Projection onto a line.
F) Reflection at a plane.
G) Reflection at a line.
H) Identity transformation.

Problem 3) (10 points)

a) Write the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ as a product of a rotation and a dilation.

b) What is the length of the vector $\vec{v} = A^{100} e_1$, where e_1 is the first basis vector?

c) In which direction does the vector $\vec{v}$ point?

d) Find a matrix B such that $B^2 = A$.

Problem 4) (10 points)

Let A be a 3×3 matrix such that $A^2 = 0$. That is, the product of A with itself is the zero matrix.

a) Verify that $\text{Im}(A)$ is a subspace of $\ker(A)$.

b) Can $\text{ran}(A) = 2$? If yes, give an example.

c) Can $\text{ran}(A) = 1$? If yes, give an example.

d) Can $\text{ran}(A) = 0$? If yes, give an example.

Problem 5) (10 points)

Let b, c be arbitrary numbers. Consider the matrix $A = \begin{bmatrix} 0 & -1 & b \\ 1 & 0 & -c \\ -b & c & 0 \end{bmatrix}$.

a) Find $\text{rref}(A)$ and find a basis for the kernel and the image of A.

b) For which b, c is the kernel one dimensional?

c) Can the kernel be two dimensional?

Problem 6) (10 points)

Consider the matrix $A = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 3 & 0 & 0 & 0 \end{bmatrix}$.

a) Use a series of elementary Gauss-Jordan row operations to find the reduced row echelon form $\text{rref}(A)$ of A. Do only one elementary operations at each step.

b) Find the rank of A.

c) Find a basis for the image of A.

d) Find a basis for the kernel of A.

Problem 7) (10 points)

Let A be a 2×2 matrix and $S = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$. We know that $B = S^{-1}AS = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find A^{2003}.

Hint. Write $B = (I_2 + C)$, note that $C^2 = 0$ and remember $(1 + x)^n = 1 + nx + .. + x^n$.

Let A be a 5×5 matrix. Suppose a finite number of elementary row operations reduces A to the following matrix $B = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.

a) Find a basis of the kernel of A.

b) Suppose the elementary row operations used in reducing A to B are the following:

i) Add row 2 to row 3.
ii) Swap row 2 and row 4.
iii) Multiple row 4 by $1/2$.
iv) Subtract row 1 from row 5.

Find a basis of the image of A.

a) Find a basis for the plane $x + 2y + z = 0$ in $\mathbf{R}^3$.

b) Find a 3×3 matrix which represents (with respect to the standard basis) a linear transformation with image the plane $x + 2y + z = 0$ and with the kernel the line $x = y = z$.