

FOURIER SERIES

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FUNCTIONS AND INNER PRODUCT? Piecewise smooth functions $f(x)$ on $[-\pi, \pi]$ form a linear space X . With an inner product in X

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$$

we can define angles, length, projections etc in the space X .

THE FOURIER BASIS. The functions $\{\cos(nx), \sin(nx), 1/\sqrt{2}\}$ form an orthonormal basis in X . This is a routine computation using the identity on the integration page.

FOURIER COEFFICIENTS. The Fourier coefficients of f are $a_0 = \langle f, 1/\sqrt{2} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)/\sqrt{2} dx$, $a_n = \langle f, \cos(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$, $b_n = \langle f, \sin(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$.

FOURIER SERIES. $f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$

ODD AND EVEN FUNCTIONS. If f is odd: $f(x) = -f(-x)$ then f has a sin series. If f is even: $f(x) = f(-x)$ then f has a cos series.

EXAMPLE. Let $f(x) = x$ on $[-\pi, \pi]$. This is an odd function ($f(-x) + f(x) = 0$) so that it has a sin series: with $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx = \frac{-1}{\pi} (x \cos(nx)/n + \sin(nx)/n^2) \Big|_{-\pi}^{\pi} = 2(-1)^{n+1}/n$, we get $x = \sum_{n=1}^{\infty} 2 \frac{(-1)^{n+1}}{n} \sin(nx)$. For example, $\pi/2 = 2(1 - 1/3 + 1/5 - 1/7 \dots)$ recovers a **formula of Leibnitz**.

EXAMPLE 2. Let $f(x) = \cos(x) + 1/7 \cos(5x)$. This **trigonometric polynomial** is already the Fourier series. The nonzero coefficients are $a_1 = 1, a_5 = 1/7$.

EXAMPLE 3. Let $f(x) = 1$ on $[-\pi/2, \pi/2]$ and $f(x) = 0$ else. This is an even function $f(-x) - f(x) = 0$ so that it has a cos series: with $a_0 = 1/(\sqrt{2}), a_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} 1 \cos(nx) dx = \frac{\sin(n\pi/2)}{\pi n} \Big|_{-\pi/2}^{\pi/2} = \frac{2(-1)^m}{\pi(2m+1)}$ if $n = 2m + 1$ is odd and 0 else. So, the series is $f(x) = 1/2 + \frac{2}{\pi} (\cos(x)/1 - \cos(3x)/3 + \cos(5x)/5 - \dots)$.

WHERE ARE FOURIER SERIES USEFUL? Examples:

- **Partial differential equations.** PDE's like the wave equation $\ddot{u} = c^2 u''$ can be solved by diagonalization (see Friday).
- **Sound** Coefficients a_k form the **frequency spectrum** of a sound f . **Filters** suppress frequencies, **equalizers** transform the Fourier space, **compressors** (i.e.MP3) select frequencies relevant to the ear.
- **Analysis:** $\sum_k a_k \sin(kx) = f(x)$ give explicit expressions for sums which would be hard to evaluate otherwise. The Leibnitz sum $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \dots$ is an example.
- **Number theory:** Example: if α is irrational, then the fact that $n\alpha \pmod{1}$ are uniformly distributed in $[0, 1]$ can be understood with Fourier theory.
- **Chaos theory:** Quite many notions in Chaos theory can be defined or analyzed using Fourier theory. Examples are mixing properties or ergodicity.
- **Quantum dynamics:** Transport properties of materials are related to spectral questions for their Hamiltonians. The relation is given by Fourier theory.
- **Crystallography:** X ray Diffraction patterns of a crystal, analyzed using Fourier theory reveal the structure of the crystal.
- **Probability theory:** The Fourier transform $\chi_X = E[e^{i\lambda X}]$ of a random variable is called **characteristic function**. Independent case: $\chi_{X+Y} = \chi_X \chi_Y$.
- **Image formats:** like JPG compress by cutting irrelevant parts in Fourier space.

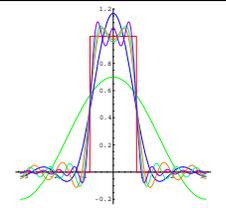
THE PARSEVAL EQUALITY. When computing the inner product of the series, we get

$$\|f\|^2 = a_0^2 + \sum_{k=1}^{\infty} a_k^2 + b_k^2.$$

EXAMPLE. $f(x) = x = 2(\sin(x) - \sin(2x)/2 + \sin(3x)/3 - \sin(4x)/4 + \dots)$ has coefficients $f_k = 2(-1)^{k+1}/k$ and so $4(1 + 1/4 + 1/9 + \dots) = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = 2\pi^2/3$ or $1 + 1/4 + 1/9 + 1/16 + 1/25 + \dots = \pi^2/6$.

APPROXIMATIONS.

If $f(x) = \sum_k b_k \cos(kx)$, then $f_n(x) = \sum_{k=1}^n b_k \cos(kx)$ is an approximation to f . Because $\|f - f_n\|^2 = \sum_{k=n+1}^{\infty} b_k^2$ goes to zero, the graphs of the functions f_n come for large n close to the graph of the function f . The picture to the left shows an approximation of a piecewise continuous even function in example 3).



SOME HISTORY. The **Greeks** approximation of planetary motion through **epicycles** was an early use of Fourier theory: $z(t) = e^{it}$ is a circle (Aristarchus system), $z(t) = e^{it} + e^{int}$ is an epicycle (Ptolemaeus system), **18'th century** Mathematicians like Euler, Lagrange, Bernoulli knew experimentally that Fourier series worked.



Fouriers claim of the convergence of the series was confirmed in the **19'th century** by Cauchy and Dirichlet. For continuous functions the sum does not need to converge everywhere. However, as the 19 year old Fejér demonstrated in his theses in 1900, the coefficients still determine the function $\sum_{k=-n}^{n-1} \frac{n-|k|}{n} f_k e^{ikx} \rightarrow f(x)$ for $n \rightarrow \infty$ if f is continuous and $f(-\pi) = f(\pi)$. Partial differential equations, like in the **theory of heat** had motivated early research in Fourier theory.



OTHER FOURIER TRANSFORMS. On a finite interval one obtains a series, on the line an integral, on finite sets, finite sums. The **discrete Fourier transformation** (DFT) is important for applications. It can be determined efficiently by the (FFT=**Fast Fourier transform**) found in 1965, reducing the n^2 steps to $n \log(n)$. Fourier theory becomes more natural in the complex, using functions e^{inx} .

Domain	Name	Synthesis	Coefficients
$\mathbf{T} = [-\pi, \pi]$	Complex Fourier series	$f(x) = \sum_k \hat{f}_k e^{ikx}$	$\hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$.
$\mathbf{R} = (-\infty, \infty)$	Complex Fourier transforms	$f(x) = \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dx$	$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$
$\mathbf{Z}_n = \{1, \dots, n\}$	Complex DFT	$f_m = \sum_{k=1}^n \hat{f}_k e^{imk2\pi/n}$	$\hat{f}_k = \frac{1}{n} \sum_{m=1}^n f_m e^{-ikm2\pi/n}$

All these transformations can be defined in dimension d . Then $k = (k_1, \dots, k_d)$ etc. are vectors. The two-dimensional discrete Fourier transform is for example useful in **image manipulation**.

COMPUTER ALGEBRA. Packages like Mathematica have the discrete Fourier transform built in Fourier[0.3,0.4,0.5] for example, gives the discrete Fourier transform of a three dimensional vector. You can perform a simple Fourier analysis yourself by listening to a sound like Play[Sin[2000 * x * Floor[7 * x]/12], {x, 0, 20}] ...

HOMEWORK:

1. Find the Fourier series of the function $f(x) = |x|$.
2. Find the Fourier series of the function $\cos^2(x) + 5 \sin(x) + 5$. You may find the double angle formula $\cos^2(x) = \frac{\cos(2x)+1}{2}$ useful.
3. Find the Fourier series of the function $f(x) = |\sin(x)|$.
4. In problem 3) you should have gotten a series

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos(2x)}{2^2-1} + \frac{\cos(4x)}{4^2-1} + \frac{\cos(6x)}{6^2-1} + \dots \right)$$

Use Parseval's identity (Fact 5.5.6 in the book) to find the value of

$$\frac{1}{(2^2-1)^2} + \frac{1}{(4^2-1)^2} + \frac{1}{(6^2-1)^2} + \dots$$