DIFFERENTIAL EQUATIONS, **Math 21b, O. Knill**

Homework: section 9.3: 6,28,34,42,44,16*,20*

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS. $Df = Tf = f'$ is a linear map on the space of smooth functions C^∞. If $p(x) = a_0 + a_1 x + ... + a_n x^n$ is a polynomial, then $p(D) = a_0 + a_1 D + ... + a_n D^n$ is a linear map on $C^\infty(R)$ too. We will see here how to find the general solution of $p(D)f = g$.

EXAMPLE. For $p(x) = x^2 - x + 6$ and $g(x) = \cos(x)$ the problem $p(D)f = g$ is the differential equation $f''(x) - f'(x) - 6f(x) = \cos(x)$. It has the solution $c_1 e^{-2x} + c_2 e^{3x} - (\sin(x) + 7\cos(x))/50$, where c_1, c_2 are arbitrary constants. How do we find these solutions?

THE IDEA. In general, a differential equation $p(D)f = g$ has many solution. For example, for $p(D) = D^3$, the equation $D^3 f = 0$ has solutions $(c_0 + c_1 x + c_2 x^2)$. The constants come from integrating three times. Integrating means applying D^{-1} but since D has as a kernel the constant functions, integration gives a one dimensional space of anti-derivatives (we can add a constant to the result and still have an anti-derivative). In order to solve $D^3 f = g$, we integrate g three times. We will generalize this idea by writing $T = p(D)$ as a product of simpler transformations which we can invert. These simpler transformations have the form $(D - \lambda)f = g$.

FINDING THE KERNEL OF A POLYNOMIAL IN D. How do we find a basis for the kernel of $Tf = f'' + 2f' + f$? The linear map T can be written as a polynomial in D which means $T = D^2 - D - 2 = (D+1)(D-2)$. The kernel of T contains the kernel of $D - 2$ which is one-dimensional and spanned by $f_1 = e^{2x}$. The kernel of $T = (D-2)(D+1)$ also contains the kernel of $D + 1$ which is spanned by $f_2 = e^{-x}$. The kernel of T is therefore two dimensional and spanned by e^{2x} and e^{-x}.

THEOREM: If $T = p(D) = D^n + a_{n-1} D^{n-1} + ... + a_1 D + a_0$ on C^∞ then $\dim(\ker(T)) = n$.

PROOF. $T = p(D) = \prod (D - \lambda_j)$, where λ_j are the roots of the polynomial p. The kernel of T contains the kernel of $D - \lambda_j$ which is spanned by $f_j(t) = e^{\lambda_j t}$. In the case when we have a factor $(D - \lambda_j)^k$ of T, then we have to consider the kernel of $(D - \lambda_j)^k$ which is $q(t)e^{\lambda t}$, where q is a polynomial of degree $k - 1$. For example, the kernel of $(D - 1)^3$ consists of all functions $(a + bt + ct^2)e^t$.

SECOND PROOF. Write this as $A\dot{g} = 0$, where A is a $n \times n$ matrix and $g = \left[f, \dot{f}, \cdots, f^{(n-1)}\right]^T$, where $f^{(k)} = D^k f$ is the k'th derivative. The linear map $T = AD$ acts on vectors of functions. If all eigenvalues λ_j of A are different (they are the same λ_j as before), then A can be diagonalized. Solving the diagonal case $BD = 0$ is easy. It has a n dimensional kernel of vectors $F = [f_1, \dots, f_n]^T$, where $f_i(t) = t$. If $B = SAS^{-1}$, and F is in the kernel of BD, then SF is in the kernel of AD.

REMARK. The result can be generalized to the case, when a_j are functions of x. Especially, $Tf = g$ has a solution, when T is of the above form. It is important that the function in front of the highest power D^n is bounded away from 0 for all t. For example $xDf(x) = e^x$ has no solution in C^∞, because we can not integrate e^x/x. An example of a ODE with variable coefficients is the **Sturm-Liouville** eigenvalue problem $T(f)(x) = a(x)f''(x) + a'(x)f'(x) + q(x)f(x) = \lambda f(x)$ like for example the Legendre differential equation $(1 - x^2)f''(x) - 2xf'(x) + n(n+1)f(x) = 0$.

BACKUP

- Equations $Tf = 0$, where $T = p(D)$ form **linear differential equations with constant coefficients** for which we want to understand the solution space. Such equations are called **homogeneous**. **Solving the equation includes finding a basis of the kernel of T.** In the above example, a general solution of $f'' + 2f' + f = 0$ can be written as $f(t) = a_1 f_1(t) + a_2 f_2(t)$. If we fix two values like $f(0), f'(0)$ or $f(0), f(1)$, the solution is unique.
- If we want to solve $Tf = g$, an **inhomogeneous equation** then T^{-1} is not unique because we have a kernel. If g is in the image of T there is at least one solution f. The general solution is then $f + \ker(T)$. For example, for $T = D^2$, which has C^∞ as its image, we can find a solution to $D^2 f = t^3$ by integrating twice: $f(t) = t^5/20$. The kernel of T consists of all linear functions $at + b$. The general solution to $D^2 = t^3$ is $at + b + t^5/20$. The integration constants parameterize actually the kernel of a linear map.

THE SYSTEM $Tf = (D - \lambda)f = g$ has the general solution $ce^{\lambda x} + e^{\lambda x}\int_0^x e^{-\lambda t} g(t)\, dt$.
The solution $f = (D - \lambda)^{-1} g$ is the sum of a function in the kernel and a special function.

THE SOLUTION OF $(D - \lambda)^k f = g$ is obtained by applying $(D - \lambda)^{-1}$ several times on g. In particular, for $g = 0$, we get: the kernel of $(D - \lambda)^k$ as $(c_0 + c_1 x + ... + c_{k-1} x^{k-1})e^{\lambda x}$.

THEOREM. The inhomogeneous $p(D)f = g$ has an n-dimensional space of solutions in $C^\infty(\mathbf{R})$.
PROOF. To solve $Tf = p(D)f = g$, we write the equation as $(D - \lambda_1)^{k_1}(D - \lambda_2)^{k_2} \cdots (D - \lambda_n)^{k_n} f = g$. Since we know how to invert each $T_j = (D - \lambda_j)^{k_j}$, we can construct the general solution by inverting one factor T_j of T one after another.
Often we can find directly a special solution f_1 of $p(D)f = g$ and get the general solution as $f_1 + f_h$, where f_h is in the n-dimensional kernel of T.

EXAMPLE 1) $Tf = e^{3x}$, where $T = D^2 - D = D(D - 1)$. We first solve $(D - 1)f = e^{3x}$. It has the solution $f_1 = ce^x + e^x \int_0^x e^{-t} e^{3t}\, dt = c_2 e^x + e^{3x}/2$. Now solve $Df = f_1$. It has the solution $c_1 + c_2 e^x + e^{3x}/6$.

EXAMPLE 2) $Tf = \sin(x)$ with $T = (D^2 - 2D + 1) = (D - 1)^2$. We see that $\cos(x)/2$ is a special solution. The kernel of $T = (D - 1)^2$ is spanned by xe^x and e^x so that the general solution is $(c_1 + c_2 x)e^x + \cos(x)/2$.

EXAMPLE 3) $Tf = x$ with $T = D^2 + 1 = (D - i)(D + i)$ has the special solution $f(x) = x$. The kernel is spanned by e^{ix} and e^{-ix} or also by $\cos(x), \sin(x)$. The general solution can be written as $c_1\cos(x) + c_2\sin(x) + x$.

EXAMPLE 4) $Tf = x$ with $T = D^4 + 2D^2 + 1 = (D - i)^2(D + i)^2$ has the special solution $f(x) = x$. The kernel is spanned by $e^{ix}, xe^{ix}, e^{-ix}, x^{-ix}$ or also by $\cos(x), \sin(x), x\cos(x), x\sin(x)$. The general solution can be written as $(c_0 + c_1 x)\cos(x) + (d_0 + d_1 x)\sin(x) + x$.

THESE EXAMPLES FORM 4 TYPICAL CASES.

CASE 1) $p(D) = (D - \lambda_1)...(D - \lambda_n)$ with real λ_i. The general solution of $p(D)f = g$ is the sum of a special solution and $c_1 e^{\lambda_1 x} + ... + c_n e^{\lambda_n x}$

CASE 2) $p(D) = (D - \lambda)^k$. The general solution is the sum of a special solution and a term $(c_0 + c_1 x + ... + c_{k-1} x^{k-1})e^{\lambda x}$

CASE 3) $p(D) = (D - \lambda)(D - \overline{\lambda})$ with $\lambda = a + ib$. The general solution is a sum of a special solution and a term $c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx)$

CASE 4) $p(D) = (D - \lambda)^k (D - \overline{\lambda})^k$ with $\lambda = a + ib$. The general solution is a sum of a special solution and $(c_0 + c_1 x + ... + c_{k-1} x^{k-1})e^{ax}\cos(bx) + (d_0 + d_1 x + ... + d_{k-1} x^{k-1})e^{ax}\sin(bx)$

We know this also from the eigenvalue problem for a matrix. We either have distinct real eigenvalues, or we have some eigenvalues with multiplicity, or we have pairs of complex conjugate eigenvalues which are distinct, or we have pairs of complex conjugate eigenvalues with some multiplicity.

CAS SOLUTION OF ODE's: Example: DSolve[f''[x] − f'[x] == Exp[3x], f[x], x]

REMARK. (informal) Operator methods are also useful for ODEs with variable coefficients. For example, $T = H - 1 = D^2 - x^2 - 1$, the **quantum harmonic oscillator**, can be written as $T = A^* A = AA^* + 2$ with a **creation operator** $A^* = (D - x)$ and **annihilation operator** $A = (D + x)$. (Hint: use the **commutation relation** $Dx - xD = 1$.) The kernel $f_0 = Ce^{-x^2/2}$ of $A = (D + x)$ is also the kernel of T and so an eigenvector of T and H. It is called the **vacuum**. If f is an eigenvector of H with $Hf = \lambda f$, then $A^* f$ is an eigenvector with eigenvalue $\lambda + 2$. Proof. Because $HA^* - A^* H = [H, A^*] = 2A^*$, we have $H(A^* f) = A^* Hf + [H, A^*]f = A^* \lambda f + 2A^* f = (\lambda + 2)(A^* f)$. We obtain all eigenvectors $f_n = A^* f_{n-1}$ of eigenvalue $1 + 2n$ by applying iteratively the creation operator A^* on the vacuum f_0. Because every function f with $\int f^2\, dx < \infty$ can be written uniquely as $f = \sum_{n=0}^\infty a_n f_n$, we can **diagonalize** H and solve $Hf = g$ with $f = \sum_n b_n/(1 + 2n) f_n$, where $g = \sum_n b_n f_n$.