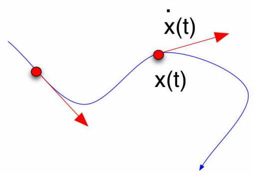
CONTINUOUS DYNAMICAL SYSTEMS I **Math 21b, O. Knill**

Homework: Section 9.1: 4,8,10,26,32,24*,46* until Tuesday

CONTINUOUS DYNAMICAL SYSTMES. A differential equation $\frac{d}{dt}\vec{x} = f(\vec{x})$ defines a dynamical system. The solutions is a curve $\vec{x}(t)$ which has the **velocity vector** $f(\vec{x}(t))$ for all t. One often writes $\dot{x}$ instead of $\frac{d}{dt}x$. So, we have the problem that we know a formula for the tangent at each point. The aim is to find a curve $\vec{x}(t)$ which starts at a given point $\vec{v} = \vec{x}(0)$.



IN ONE DIMENSION. A system $\dot{x} = g(x,t)$ is the general differential equation in one dimensions. Examples:

- If $\boxed{\dot{x} = g(t)}$, then $x(t) = \int_0^t g(t)\, dt$. Example: $\dot{x} = \sin(t), x(0) = 0$ has the solution $x(t) = \cos(t) - 1$.
- If $\boxed{\dot{x} = h(x)}$, then $dx/h(x) = dt$ and so $t = \int_0^x dx/h(x) = H(x)$ so that $x(t) = H^{-1}(t)$. Example: $\dot{x} = \frac{1}{\cos(x)}$ with $x(0) = 0$ gives $dx\cos(x) = dt$ and after integration $\sin(x) = t + C$ so that $x(t) = \arcsin(t + C)$. From $x(0) = 0$ we get $C = \pi/2$.
- If $\boxed{\dot{x} = g(t)/h(x)}$, then $H(x) = \int_0^x h(x)\, dx = \int_0^t g(t)\, dt = G(t)$ so that $x(t) = H^{-1}(G(t))$. Example: $\dot{x} = \sin(t)/x^2, x(0) = 0$ gives $dx x^2 = \sin(t)dt$ and after integration $x^3/3 = -\cos(t) + C$ so that $x(t) = (3C - 3\cos(t))^{1/3}$. From $x(0) = 0$ we obtain $C = 1$.

Remarks:
1) In general, we have no closed form solutions in terms of known functions. The solution $x(t) = \int_0^t e^{-t^2}\, dt$ of $\dot{x} = e^{-t^2}$ for example can not be expressed in terms of functions $\exp, \sin, \log, \sqrt{\cdot}$ etc but it can be solved using Taylor series: because $e^{-t^2} = 1 - t^2 + t^4/2! - t^6/3! + \dots$ taking coefficient wise the anti-derivatives gives: $x(t) = t - t^3/3 + t^4/(32!) - t^7/(73!) + \dots$.
2) The system $\dot{x} = g(x,t)$ can be written in the form $\vec{x} = f(\vec{x})$ with $\vec{x} = (x,t)$. $\frac{d}{dt}\begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} g(x,t) \\ 1 \end{bmatrix}$.

ONE DIMENSIONAL LINEAR DIFFERENTIAL EQUATIONS. The system $\boxed{\dot{x} = \lambda x}$ has the solution $\boxed{x(t) = e^{\lambda t}x(0)}$. This differential equation appears

- as **population models** with $\lambda > 0$: birth rate of the population is proportional to its size.
- as a model for **radioactive decay** with $\lambda < 0$: the rate of decay is proportional to the number of atoms.

LINEAR DIFFERENTIAL EQUATIONS IN A NUTSHELL: Linear dynamical systems have the form $\dot{x} = Ax$, where A is a matrix. $\vec{0}$ is an **equilibrium point**: if $\vec{x}(0) = \vec{0}$, then $\vec{x}(t) = \vec{0}$ for all t. In general, we look for a solution $\vec{x}(t)$ for a given initial point $\vec{x}(0) = \vec{v}$. Here are three different ways to express the closed solution:

- Linear differential equations can be solved as in one dimensions: the general solution to $\dot{x} = Ax, \vec{x}(0) = \vec{v}$ is $\boxed{x(t) = e^{At}\vec{v} = (1 + At + A^2t^2/2! + \dots)\vec{v}}$, because $\dot{x}(t) = A + 2A^2t/2! + \dots = A(1 + At + A^2t^2/2! + \dots)\vec{v} = Ae^{At}\vec{v} = Ax(t)$. However, this solution is not very useful and is also computationally not convenient.
- If $B = S^{-1}AS$ is diagonal with the eigenvalues $\lambda_j = a_j + ib_j$, then $y = S^{-1}x$ satisfies $y(t) = e^{Bt}$ and therefore $\boxed{y_j(t) = e^{\lambda_j t}y_j(0) = e^{a_j t}e^{ib_j t}y_j(0)}$. The solutions in the original coordinates are $x(t) = Sy(t)$.
- If $\vec{v}_i$ are the eigenvectors to the eigenvalues λ_i, and $\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$, then $\boxed{\vec{x}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + \dots + c_ne^{\lambda_n t}\vec{v}_n}$ is a closed formula for the solution of $\frac{d}{dt}\vec{x} = A\vec{x}, \vec{x}(0) = \vec{v}$.

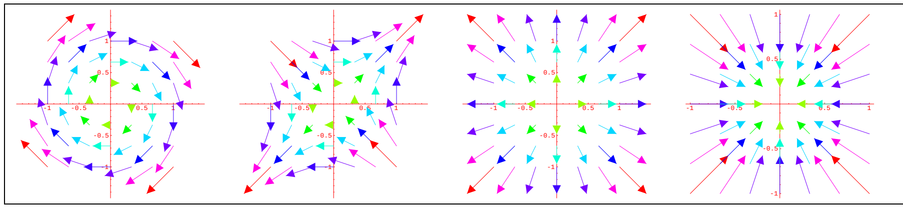
EXAMPLE. Find a closed formula for the solution of the system

$$\begin{aligned} \dot{x}_1 &= x_1 + 2x_2 \\ \dot{x}_2 &= 4x_1 + 3x_2 \end{aligned}$$

with $\vec{x}(0) = \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The system can be written as $\dot{x} = Ax$ with $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. The matrix A has the eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to the eigenvalue -1 and the eigenvector $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ to the eigenvalue 5.

Because $A\vec{v}_1 = -\vec{v}_1$, we have $\vec{v}_1(t) = e^{-t}\vec{v}$. Because $A\vec{v}_2 = 5\vec{v}_1$, we have $\vec{v}_2(t) = e^{5t}\vec{v}$. The vector $\vec{v}$ can be written as a linear-combination of $\vec{v}_1$ and $\vec{v}_2$: $\vec{v} = \frac{1}{3}\vec{v}_2 + \frac{2}{3}\vec{v}_1$. Therefore, $\vec{x}(t) = \frac{1}{3}e^{5t}\vec{v}_2 + \frac{2}{3}e^{-t}\vec{v}_1$.

PHASE PORTRAITS. For differential equations $\dot{x} = f(x)$ in two dimensions one can **draw the vector field** $x \mapsto f(x)$. The solution curve $x(t)$ is tangent to the vector $f(x(t))$ everywhere. The phase portraits together with some solution curves reveal much about the system. Experiment with the Java applet on the web-site! Examples:



UNDERSTANDING A DIFFERENTIAL EQUATION. The closed form solution like $x(t) = e^{At}x(0)$ for $\dot{x} = Ax$ is actually quite useless. One wants to understand the solution quantitatively. Questions one wants to answer are: what happens in the long term? Is the origin stable, are there periodic solutions. Can one decompose the system into simpler subsystems? We will see that **diagonalisation** allows to **understand the system:** by decomposing it into one-dimensional linear systems, which can be analyzed separately. In general, "understanding" can mean different things:

Plotting phase portraits. Computing solutions numerically and estimate the error. Finding special solutions. Predicting the shape of some orbits. Finding regions which are invariant.	Finding special closed form solutions $x(t)$. Finding a power series $x(t) = \sum_n a_n t^n$ in t. Finding quantities which are unchanged along the flow (called "Integrals"). Finding quantities which increase along the flow (called "Lyapunov functions").

LINEAR STABILITY. A linear dynamical system $\dot{x} = Ax$ with diagonalizable A is linearly stable if and only if $a_j = \mathrm{Re}(\lambda_j) < 0$ for all eigenvalues λ_j of A.

PROOF. We see that from the explicit solutions $y_j(t) = e^{a_j t}e^{ib_j t}y_j(0)$ in the basis consisting of eigenvectors. Now, $y(t) \to 0$ if and only if $a_j < 0$ for all j and $x(t) = Sy(t) \to 0$ if and only if $y(t) \to 0$.

RELATION WITH DISCRETE TIME SYSTEMS. From $\dot{x} = Ax$, we obtain $x(t+1) = Bx(t)$, with the matrix $B = e^A$. The eigenvalues of B are $\mu_j = e^{\lambda_j}$. Now $|\mu_j| < 1$ if and only if $\mathrm{Re}\lambda_j < 0$. The criterium for linear stability of discrete dynamical systems is compatible with the criterium for linear stability of $\dot{x} = Ax$.

EXAMPLE 1. The system $\dot{x} = y, \dot{y} = -x$ can in vector form $v = (x,y)$ be written as $\dot{v} = Av$, with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The matrix A has the eigenvalues $i, -i$. After a coordinate transformation $w = S^{-1}v$ we get with $w = (a,b)$ the differential equations $\dot{a} = ia, \dot{b} = -ib$ which has the solutions $a(t) = e^{it}a(0), b(t) = e^{-it}b(0)$. The original coordates satisfy $x(t) = \cos(t)x(0) - \sin(t)y(0)$, $y(t) = \sin(t)x(0) + \cos(t)y(0)$. Indeed e^{At} is a rotation in the plane.