REVIEW HOURLY I **Math 21b, O. Knill**

DEFINITIONS.

- **Linear subspace** $\vec{0} \in X$, $\vec{x}, \vec{y} \in X, \lambda \in \mathbf{R} \Rightarrow \vec{x} + \vec{y} \in X$, $\lambda \vec{x} \in X$.
- **Matrix** A is a $n \times m$ matrix, it has m columns and n rows, maps $\mathbf{R}^m$ to $\mathbf{R}^n$.
- **Square matrix** $n \times n$ matrix, maps $\mathbf{R}^n$ to $\mathbf{R}^n$.
- **Vector** $n \times 1$ matrix = column vector, $1 \times n$ matrix = row vector.
- **Linear transformation** $T : \mathbf{R}^n \to \mathbf{R}^m$, $\vec{x} \mapsto A\vec{x}$, $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}), T(\lambda\vec{x}) = \lambda T(\vec{x})$.
- **Column vector** of A are images of standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$.
- **Linear system of equations** $A\vec{x} = \vec{b}$, n equations, m unknowns.
- **Consistent system** $A\vec{x} = \vec{b}$: for every $\vec{b}$ there is at least one solution $\vec{x}$.
- **Vector form of linear equation** $x_1\vec{v}_1 + \ldots + x_n\vec{v}_n = \vec{b}$, $\vec{v}_i$ columns of A.
- **Matrix form of linear equation** $\vec{w}_i \cdot \vec{x} = b_i$, $\vec{w}_i$ rows of A.
- **Augmented matrix of** $A\vec{x} = \vec{b}$ is the matrix $[A|b]$ which has one column more as A.
- **Coefficient matrix of** $A\vec{x} = \vec{b}$ is the matrix A.
- **Matrix multiplication** $[AB]_{ij} = \sum_k A_{ik}B_{kj}$, dot product of i-th row of A with j'th column of B.
- **Gauss-Jordan elimination** $A \to \text{rref}(A)$ in row reduced echelon form.
- **Gauss-Jordan elimination steps:** swapping rows, scaling rows, adding rows to other rows.
- **Row reduced echelon form**: every nonzero row has leading 1, columns with leading 1 are 0 away from leading 1, every row with leading 1 has every rows above with leading 1 to the left.
- **Pivot column** column with leading 1 in rref(A).
- **Redundant column** column with no leading 1 in rref(A).
- **Rank of matrix** A. Number of leading 1 in rref(A). It is equal to dim(im(A)).
- **Nullety of matrix** A. Is defined as dim(ker(A)).
- **Kernel of linear transformation** $\{\vec{x} \in \mathbf{R}^n, A\vec{x} = \vec{0}\}$.
- **Image of linear transformation** $\{A\vec{x}, \vec{x} \in \mathbf{R}^n\}$.
- **Inverse** Linear transformation satisfying $S(T(x)) = x = T(S(x))$. Corresponding matrix $B = A^{-1}$.
- **Rotation in plane** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$, counterclockwise rotation by angle α.
- **Dilation in plane** $\vec{x} \mapsto \lambda\vec{x}$, also called scaling.
- **Rotation-Dilation** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Scale by $\sqrt{a^2 + b^2}$, rotate by $\arctan(b/a)$.
- **Horizontal and vertical shear** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$.
- **Shear** T leaves $\vec{v}$ invariant and $A\vec{x} - \vec{x}$ parallel to $\vec{v}$. Shear-Check: all $A\vec{x} - \vec{x}$ are parallel.
- **Reflection at line** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$.
- **Projection onto line containing unit vector** $T(\vec{x}) = (\vec{x} \cdot \vec{v})\vec{v}$.
- $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **span** X: Every $\vec{x} \in X$ can be written as $\vec{x} = a_1\vec{v}_1 + ... + a_n\vec{v}_n$.
- $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **linear independent** X: $\vec{0} = \sum_i a_i\vec{v}_i \Rightarrow a_1 = \ldots = a_n = 0$.
- $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **basis in** X: linear independent in X and span X.
- **dimension of linear space** X number of elements in a basis of X.
- $\mathcal{B}$ **coordinates** $[\vec{v}]_{\mathcal{B}} = S^{-1}\vec{v}$, where $S = [\vec{v}_1, \dots, \vec{v}_n]$ contains basis vectors $\vec{v}_i$ as columns.
- $\mathcal{B}$ **matrix of** T The matrix is $B = S^{-1}AS$.

RESULTS.

- **Linear transformations**. T is linear: $T(\vec{0}) = \vec{0}, T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}), T(\lambda\vec{x}) = \lambda T(\vec{x})$
- **Solution**. A linear system of equations has either exactly 1, no or infinitely many solutions.
- **Dimension formula**. dim(ker(A)) + dim(im(A)) = m, where A is $n \times m$ matrix.
- **Behavior of kernel under elimination** kernel stays invariant under Gauss-Jordan elimination.
- **Behavior of image under elimination** image in general changes during Gauss-Jordan elimination.
- **Basis of image of** A pivot columns of A form a basis of the image of A.
- **Basis of kernel of** A introduce free variables for each non-Pivot column of A.
- **Inverse of** 2×2 **matrix** switch diagonal elements, change sign of wings and divide by determinant.
- **Kernel of composition** kernel of A is contained in the kernel of BA.
- **Image of composition** image of BA is contained in the image of B.
- **Matrix algebra** $(AB)^{-1} = B^{-1}A^{-1}$, $A(B + C) = AB + AC$, etc. Note: $AB \neq BA$ in general.
- **A invertible** $\Leftrightarrow$ rref(A) = I_n $\Leftrightarrow$ columns form basis $\Leftrightarrow$ rank(A) = n, $\Leftrightarrow$ nullity(A) = 0 .

PROBLEM You know $B = \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find the nullity and rank of A. Can you find the kernel of A? Can you find the image of A.

SOLUTION. We can find the kernel but not the image. The dimensions of the image and kernel are 2 (2 pivot, 2 nonpivot columns). The linear system to B is $x + 2y + 5z = 0, z + 3u = 0$. Solving gives $u = t, z = -3t, y = s, x = -2s - 5t$, so that a general kernel element is $t[-5, 0, -3, 1] + s[-2, 1, 0, 0]$.

PROBLEM Given the basis $\vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ in the plane. a) Find the coordinates of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in this basis. b) Given a vector with coordinates $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in that basis, find the coordinates in the standard basis.

SOLUTION. $S = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$ has the inverse $S^{-1} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$. a) The vector has new coordinates $S^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ 7 \end{bmatrix}$, b) The vector has the standard coordinates $S\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$.

PROBLEM Let A be a shear along a line L. Find $\ker(A - I_2)$, $\text{im}(A - I_2)$ and $(A - I_2)^2$.

SOLUTION. Shears have the property that $(A - I)x = Ax - x$ is parallel to L, therefore $\text{im}(A - I)$ is L. The kernel of $A - I$ consists of vectors $Ax = x$. Because every $v \in L$ has this property, the kernel is L too. Both have dimension 1. We have $(A - 1)^2 = 0$ because the image of $A - 1$ is a subset of the kernel.

PROBLEM Let L be the line spanned by $\vec{v} = (1/2, 1/2, 0)$ and let T be the counterclockwise rotation about an angle $\pi/2$ around L (this means $\pi/2$ clockwise if you look from $\vec{v}$ to the origin). Find the matrix A.

SOLUTION. Draw a good picture. $\vec{e}_1$ goes to $[1/2, 1/2, 1/\sqrt{2}]$, $\vec{e}_2$ goes to $[1/2, 1/2, -1/\sqrt{2}]$, $\vec{e}_3$ goes to $[-1/\sqrt{2}, 1/\sqrt{2}, 0]$. These are the columns of A, so that $A = \begin{bmatrix} 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$.

PROBLEM. Let A be a 3×3 matrix satisfying $A^2 = 0$. Show that the image of A is a subset of the kernel of A and determine all possible values for rank(A). Give an example.

SOLUTION. If x is in the image then $x = Ay$ and $Ax = AAy = 0$ so that x is in the kernel. A is not invertible and can not have rank 3. It can be the 0 matrix with rank 0. It can have rank 1: take 0 in the first two columns and e_1 in the last. It can not have rank 2 because the dimension of the kernel would be 1 and could not contain the kernel.

PROBLEM. Let $A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$ Find B satisfying $B^2 = A$, determine rank(B) and B^{17}.

SOLUTION. A is a rotation-dilation, a composition of a rotation by $\pi/2$ and dilation by 4. Take B as a rotation dilation with angle $\pi/4$ and dilation factor 2. The rank of B is 2 because if it were smaller, then also the rank of A were smaller. B^{17} is a rotation dilation with angle $17\pi/2 \sim \pi/2$ and dilation factor 2^{17}.

PROBLEM. If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbf{R}^n$ are linearly independent, are $\vec{w}_1 = \vec{v}_1, \vec{w}_2 = \vec{v}_1 + \vec{v}_2, \vec{w}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ also linearly independent?

SOLUTION. Yes: the linear map which maps the span of $\vec{v}_i$ into the span of $\vec{w}_i$ is invertible: it has the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

PROBLEM. Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix}$.

SOLUTION. Deleting the first row from each others shows that the rank is 2.