

The Lorentz Boost

Math21b, Oliver Knill

This background information is not part of the course. The relation with special relativity might be fun to know about. We will use the functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \sinh(x) = \frac{e^x - e^{-x}}{2}$$

on this page.

LORENTZ BOOST. The linear transformation of the plane given by the matrix

$$A = \begin{bmatrix} \cosh(\phi) & \sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{bmatrix}$$

is called the **Lorentz boost**. The angle ϕ depends on the velocity v . The corresponding transformation

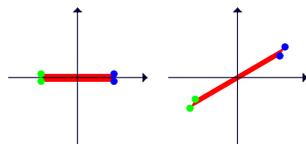
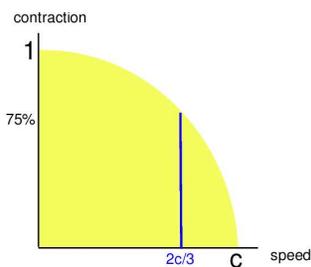
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto A \begin{bmatrix} x \\ y \end{bmatrix} \text{ with } y = ct \text{ is important in relativistic physics.}$$

PHYSICAL INTERPRETATION. In classical mechanics, when a particle travels with velocity v on the line, its new position satisfies $\tilde{x} = x + tv$. The Galileo transformation is $\begin{bmatrix} x \\ ct \end{bmatrix} \mapsto \begin{bmatrix} x + tv \\ ct \end{bmatrix}$. According to special relativity, this is only an approximation. In reality, the motion is described by a **linear transformation** $\begin{bmatrix} x \\ ct \end{bmatrix} \mapsto A \begin{bmatrix} x \\ ct \end{bmatrix}$, where A is the above matrix and where the angle ϕ is related to v by the formula $\tanh(\phi) = v/c$. Trigonometric identities give $\sinh(\phi) = (v/c)/\gamma$, $\cosh(\phi) = 1/\gamma$, where $\gamma = \sqrt{1 - v^2/c^2}$. The linear transformation is $A(x, ct) = ((x + vt)/\gamma, t + (v/c^2)/\gamma x)$. For small velocities v , where the value of γ is close to 1 and v/c^2 is close to zero, $A(x, ct)$ is close to $(x + vt, ct)$.

LORENTZ CONTRACTION. If we displace a ruler $[a, b]$ with velocity v then its end points are not $[a + tv, b + tv]$ as Newtonian mechanics would tell but $[(a + tv)/\gamma, (b + tv)/\gamma]$. The ruler is by a factor $1/\gamma$ larger, when we measure it in the moving coordinate system, where the ruler rests. The constant γ is called the **Lorentz contraction factor**.

Example: $v = 2c/3$, where c is the speed of light, the contraction is 75 percent.

In the fixed coordinate system, the two end points of the ruler have a different time. If a light signal would be sent out simultaneously at both ends, then this signal would reach the origin at different times. The one to the left earlier than the one to the right. The end point to the left is "younger".)



Ruler in (x, ct) plane. Left: system which travels with the ruler, right: fixed system in which the ruler moves.