MATH 21B REVIEW
Part II

Harvard, Jan 13, 2004

Content

Methodology of this lecture

Slides to review the concepts

It's the PowerPoint, Stupid

31 December 2003
by Marc Zeedar
Contributing Columnist

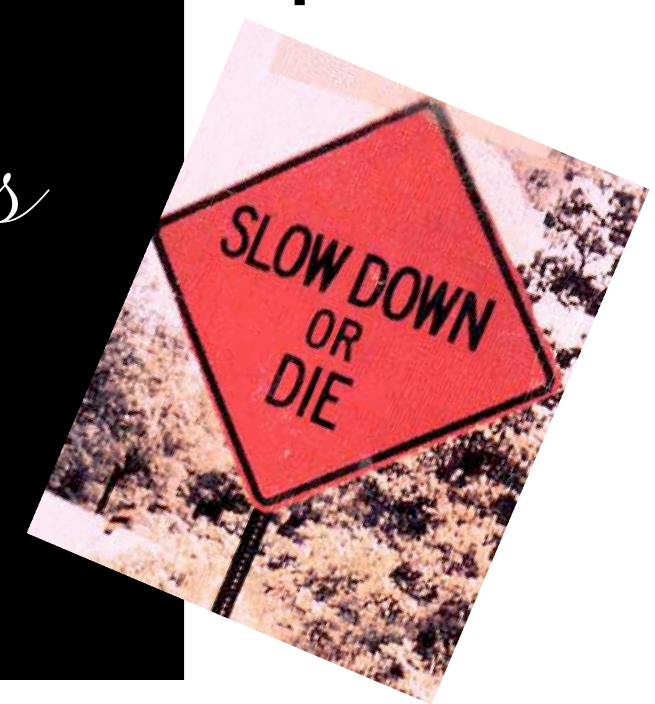
print format | email this story | talk

The *New York Times* recently ran an article titled PowerPoint Makes You Dumb. In it, the author cites sources who blame the Columbia space shuttle disaster on Microsoft's presentation software. You see, apparently those guys at NASA used a PowerPoint slide to "explain"

80 slides

7 problems

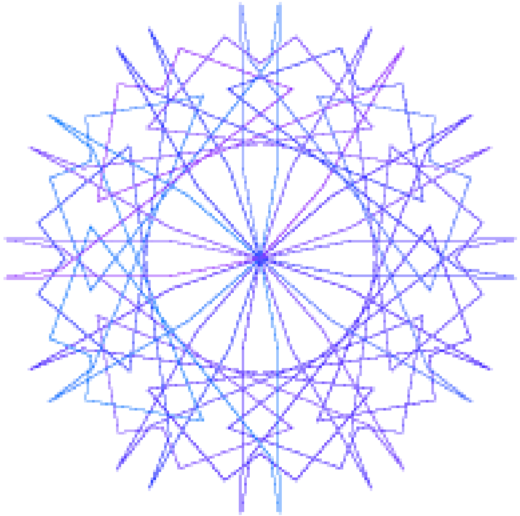
Blackboard problems



I) Preliminaries

- Diagonalization
- Complex Numbers
- Linear spaces, linear transformations
- Differential operators

Diagonalization



Diagonalization of a matrix A is possible if:

Orthonormal eigenbasis

- A is a symmetric matrix
- All eigenvalues of A are different

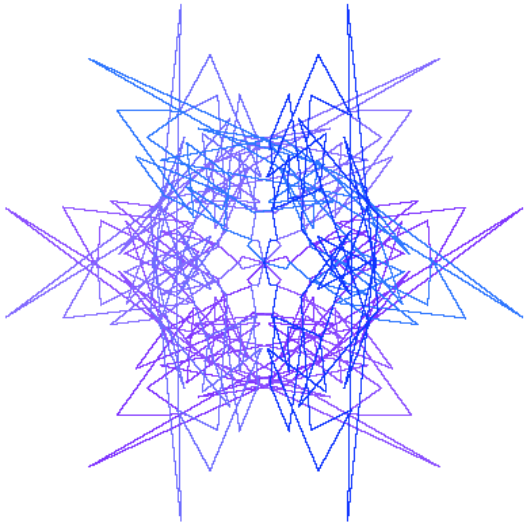
Prototype of nondiagonalizable matrix :

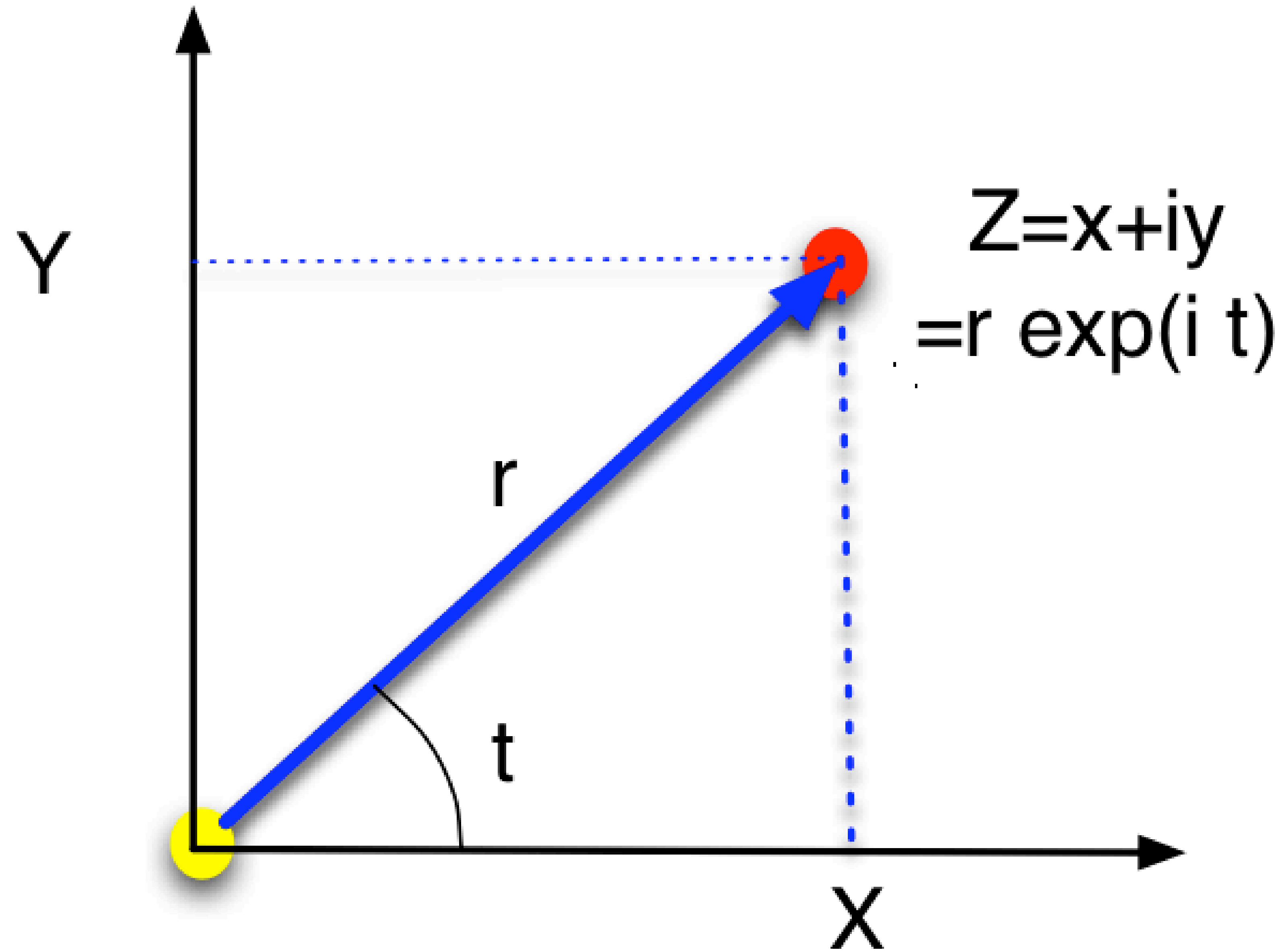
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Jordan Normal Form

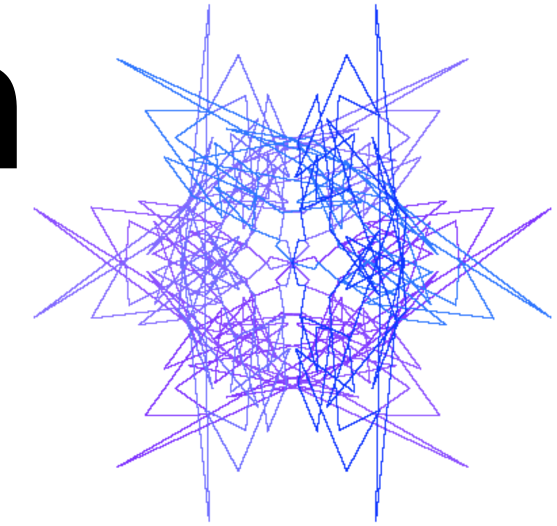
$$\begin{bmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

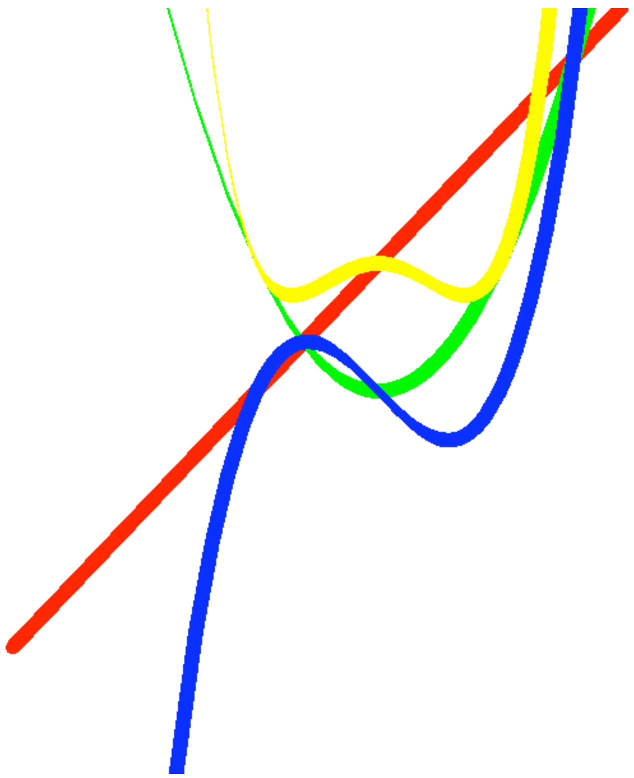
Complex Numbers





Fundamental Theorem of algebra

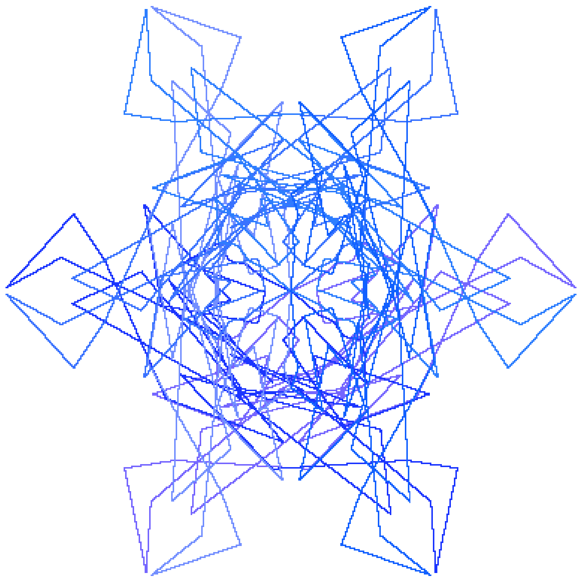


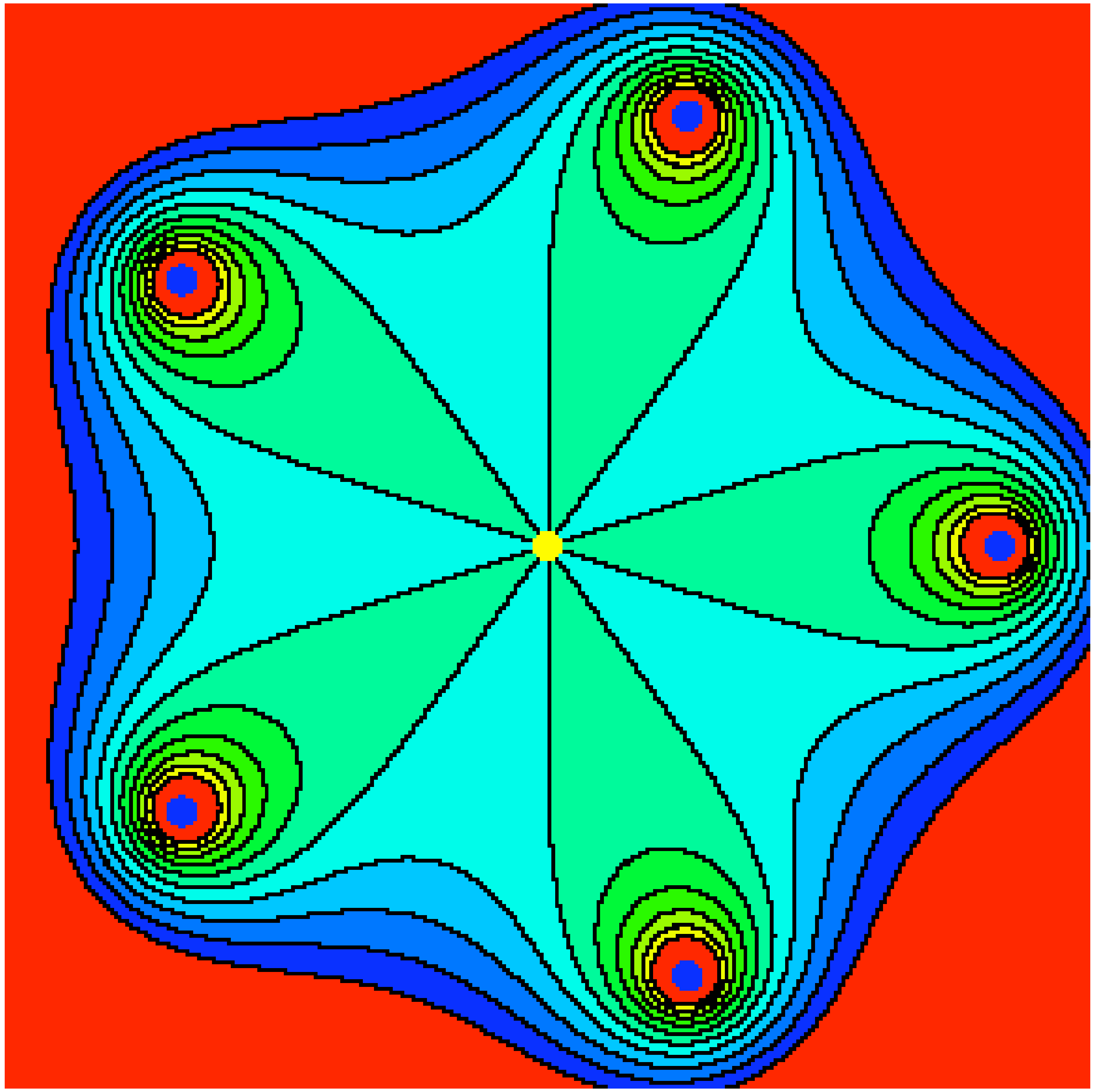


A polynomial of degree n has exactly n roots.

- If a+ib is a root, then a-ib is a root too.
- If n is odd, there is at least one real root.

Example: Roots of One



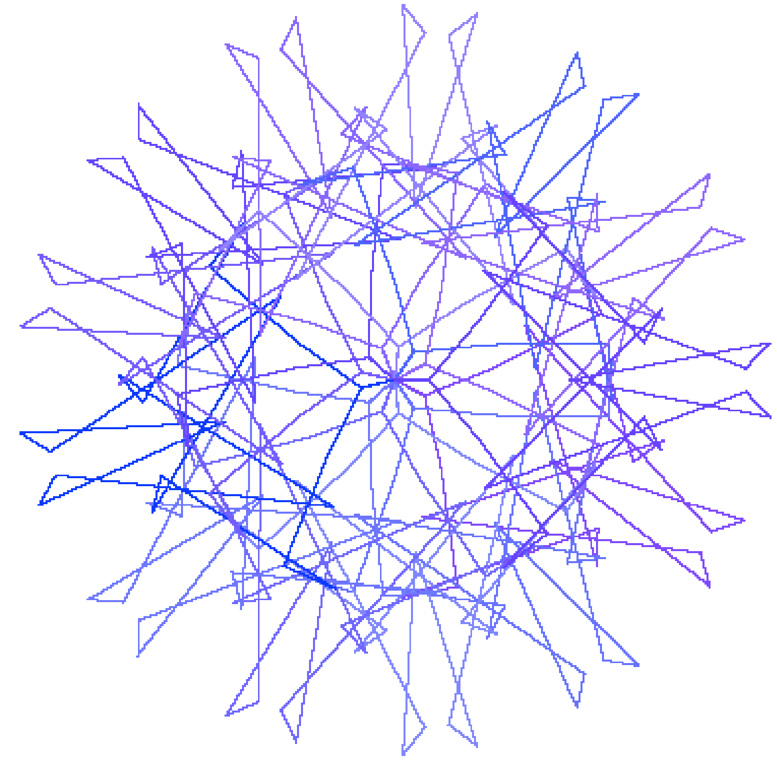


Example:
permutation matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$p(\lambda) = (\lambda^n - 1) = (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$$

Linear Spaces



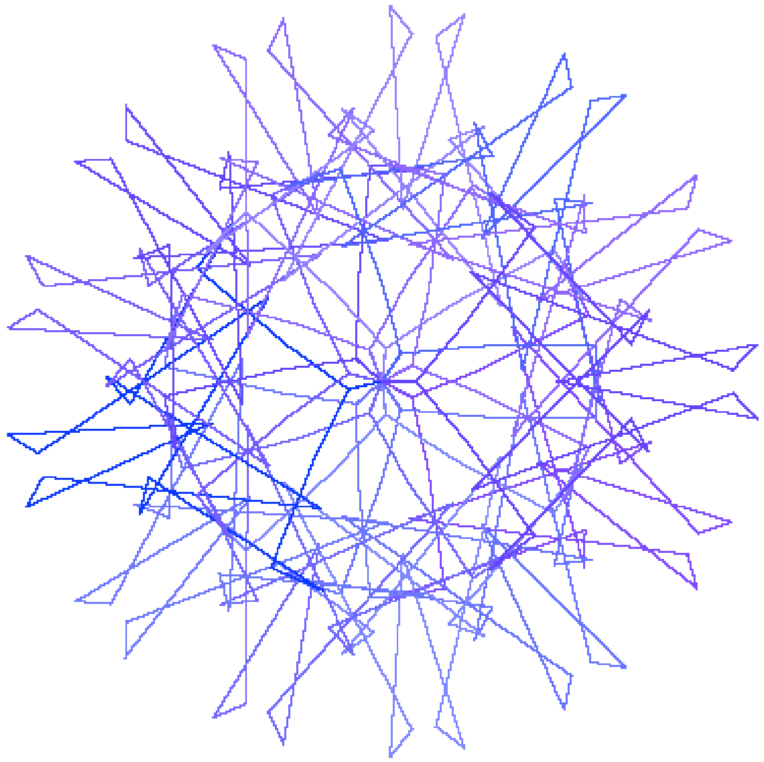
In a linear space, we can add and scale. It contains a zero element.

$\mathbf{R}^n$	$\mathbf{M}(\mathbf{R}, n)$
$\mathrm{C}^\infty(\mathbf{R})$	$\mathrm{C}^\infty([-\pi, \pi])$

To check whether a subset X of one of these spaces is a linear space, we check:

x+y is in X	rx is in X	0 is in X

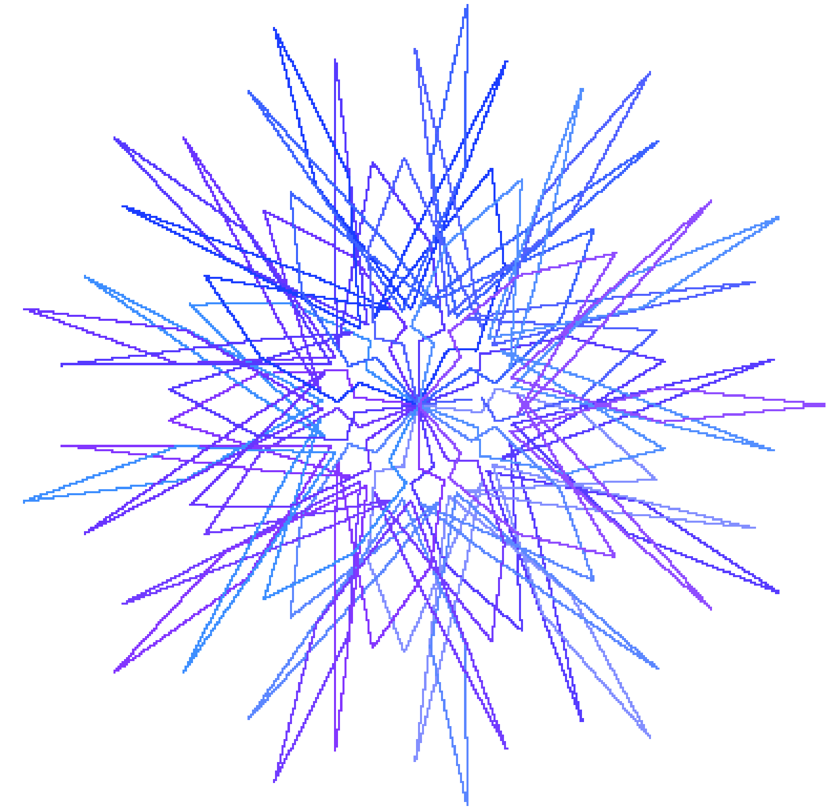
Linear Maps



To check wether a map between linear spaces is linear, we have to check:

$$T(x+y) = T(x)+T(y), \quad T(rx)=r\,T(x), \quad T(0)=0$$

Differential operators



Df=f’

p(x) polynomial

T=p(D) differential operator

T f = g differential equation

Fundamental theorem of calculus:

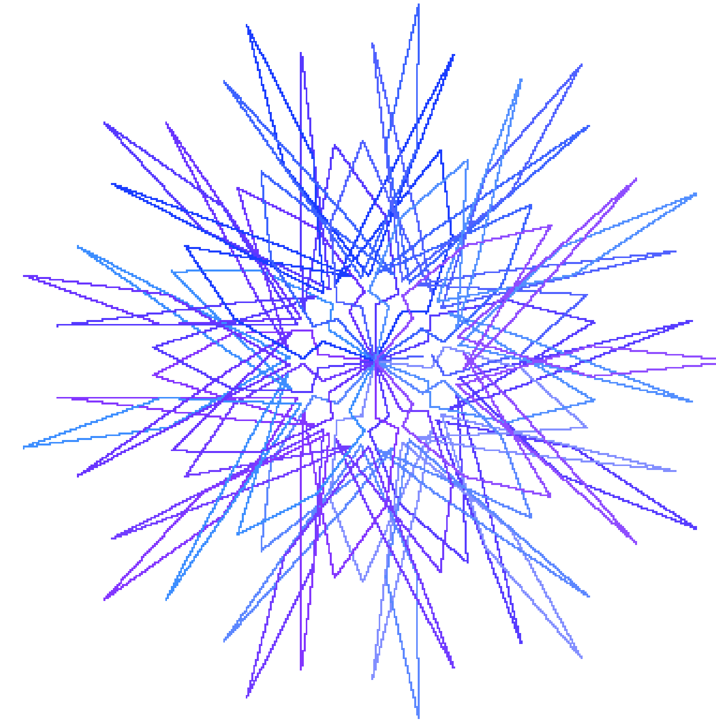
$$p(D) = (D - \lambda_1)(D - \lambda_2)...(D - \lambda_n)$$

Inverse

$$D^{-1} f(t) = \int_0^t f(t)\, dt + C$$

$$(D - \lambda)^{-1} f(t) = e^{\lambda t} \left(\int_0^t e^{-\lambda t} f(t)\, dt + C \right)$$

Example:



$$T(f) = D^2 f + 10 \ Df \ + 21$$
$$= (D+3)\ (D+7)$$

{ T(f) =0 } the kernel,
is a linear space
spanned by e^{-3t} and e^{-7t}

$f'' + 10 f' + 21 = 0$

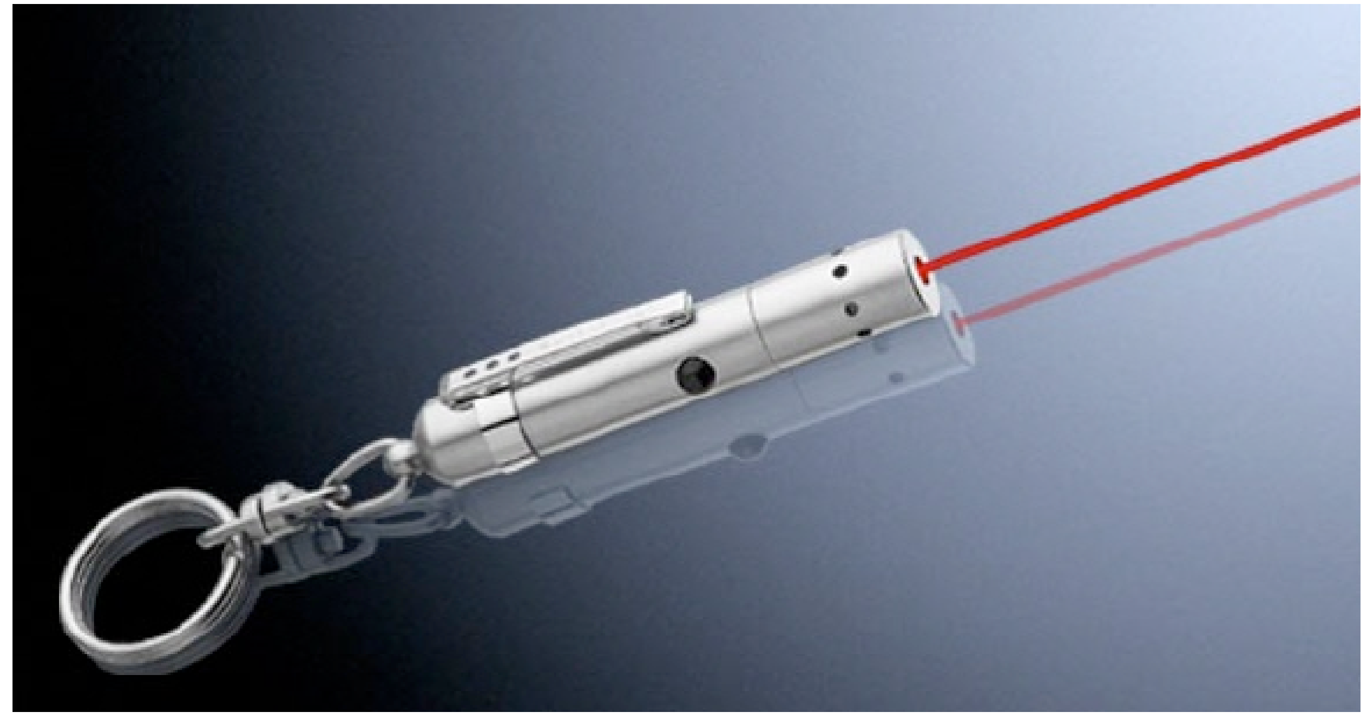
{ T(f) =g } not a linear
space if g is not zero.

The solution set is formed by
adding a particular solution to
the kernel. See later.

$f'' + 2f' + 10 = g$

Quizz coming up!

Win laser pointer



Structure of the quizz:

2)
3)
5)
7)
11)

True

2)
3)
5)
7)
11)

Shout:

21 14

I used to have Swiss Army knife prizes. Will soon also laser pointers be banned?

Posted 1/4/2005 12:41 PM Updated 1/5/2005 9:48 AM

N.J. man charged with aiming laser at aircraft

By Alan Levin, USA TODAY

A New Jersey man was charged Tuesday under federal anti-terrorism laws with shining a laser beam at a charter jet flying over his home, temporarily distracting the pilots.

If convicted of anti-terrorism violations, David Banach of Parsippany, N.J., faces up to 25 years in prison and fines of up to $500,000.

By Mike Derer, AP

David Banach, 38, is the first person charged in a rash of recent

Again: here are the rules for the upcoming multiple choice challenge: 10 questions. Encode the answer:

If your choice is:

2) 3) 5) 7) 11)

True

2) 3) 5) 7) 11)

then shout:

21 14

Linear space or not?

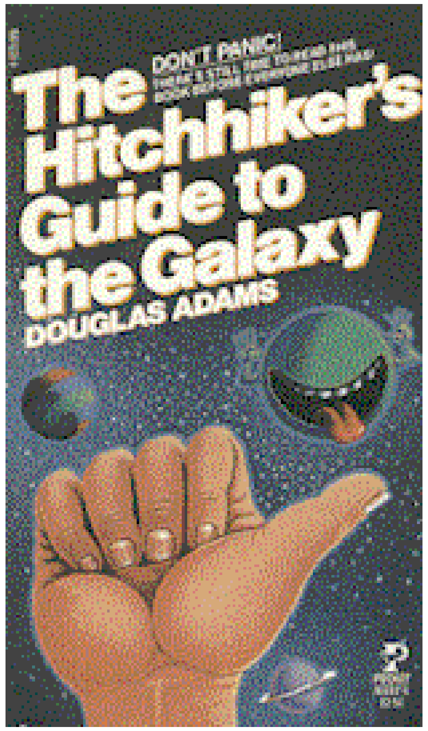
2) All smooth functions on $[0, 2\pi]$ satisfying $\int_0^{2\pi} f(x)\, dx = 0$.
3) All smooth functions satisfying $f(10) = 0$
5) All Smooth functions on the line satisfying $f'(10) = 10$.
7) All symmetric 2×2 matrices.
11) All polynomials of degree 10.

Linear map or not?

2) $T(f)(x) = x^2 f(x)$.
3) $T(f)(x) = f''(x)$.
5) $T(f)(x) = f(1)^2 + f(x)$.
7) $T(f)(x) = f(5)$.
11) $T(f)(x) = f(x)f'(x)$.

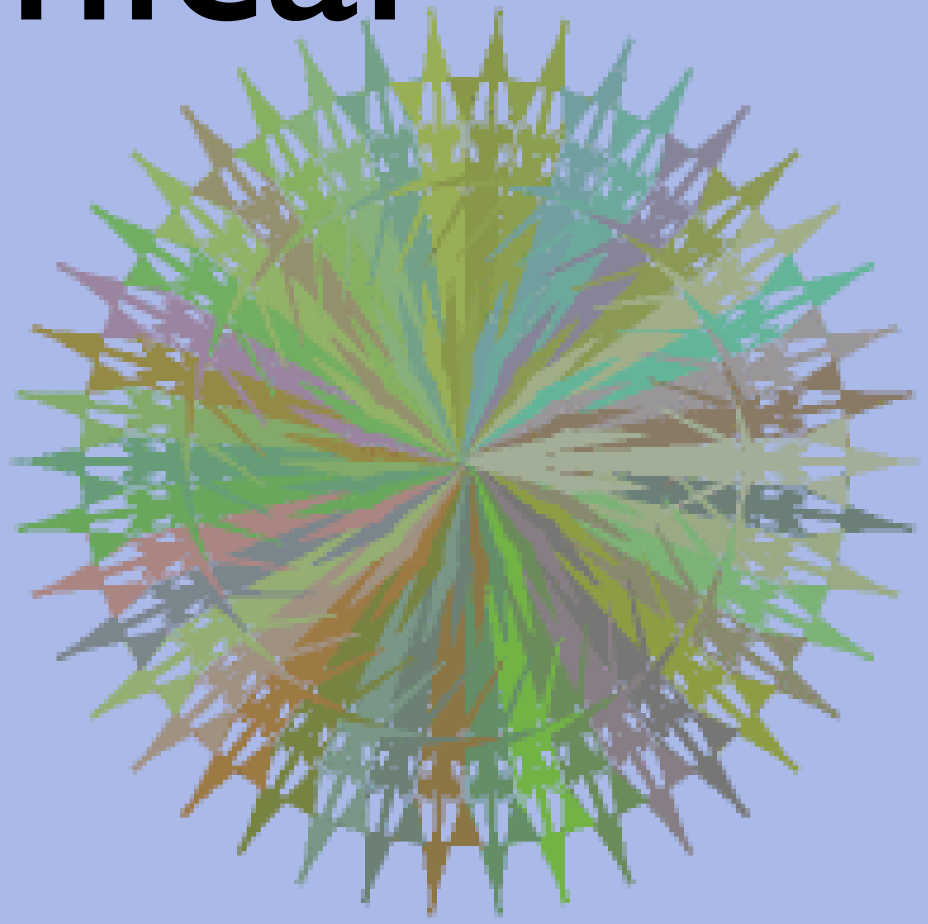
42

The answer to the Ultimate Question of Life, the Universe and Everything, given by the supercomputer 'Deep Thought' to a group of mice, is "forty-two".



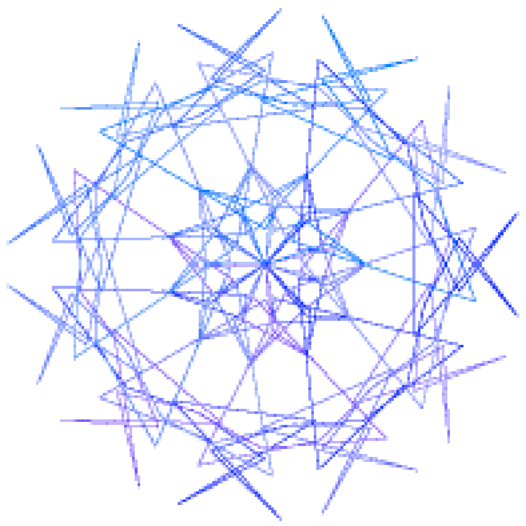
"Forty-two!" yelled Loonquawl. "Is that all you've got to show for seven and a half million years' work?" "I checked it very thoroughly," said the computer, "and that quite definitely is the answer. I think the problem, to be quite honest with you, is that you've never actually known what the question is."

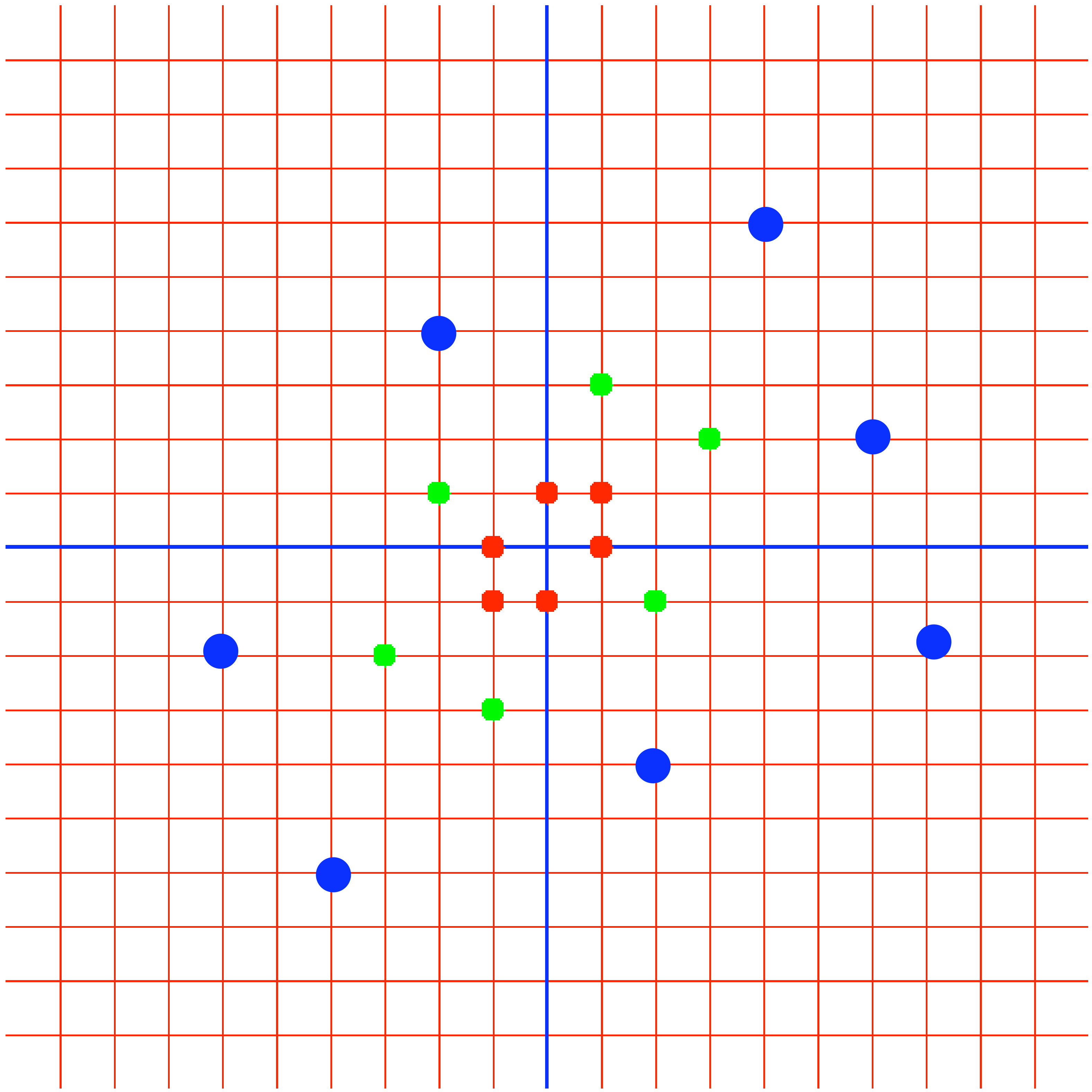
II) Discrete Dynamical Systems



- Solving initial value problems
- Analyse phase space
- Find out about stability

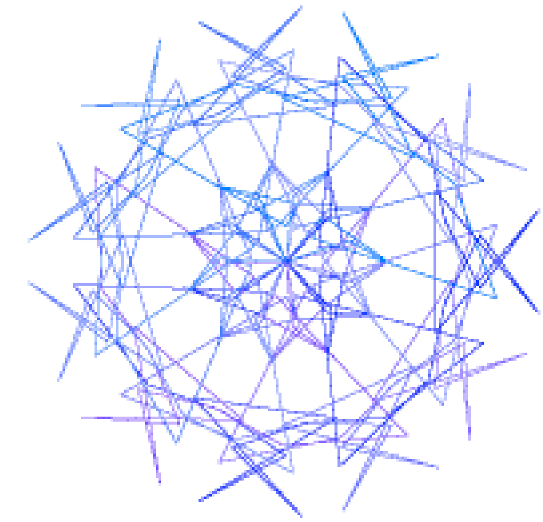
The dynamics

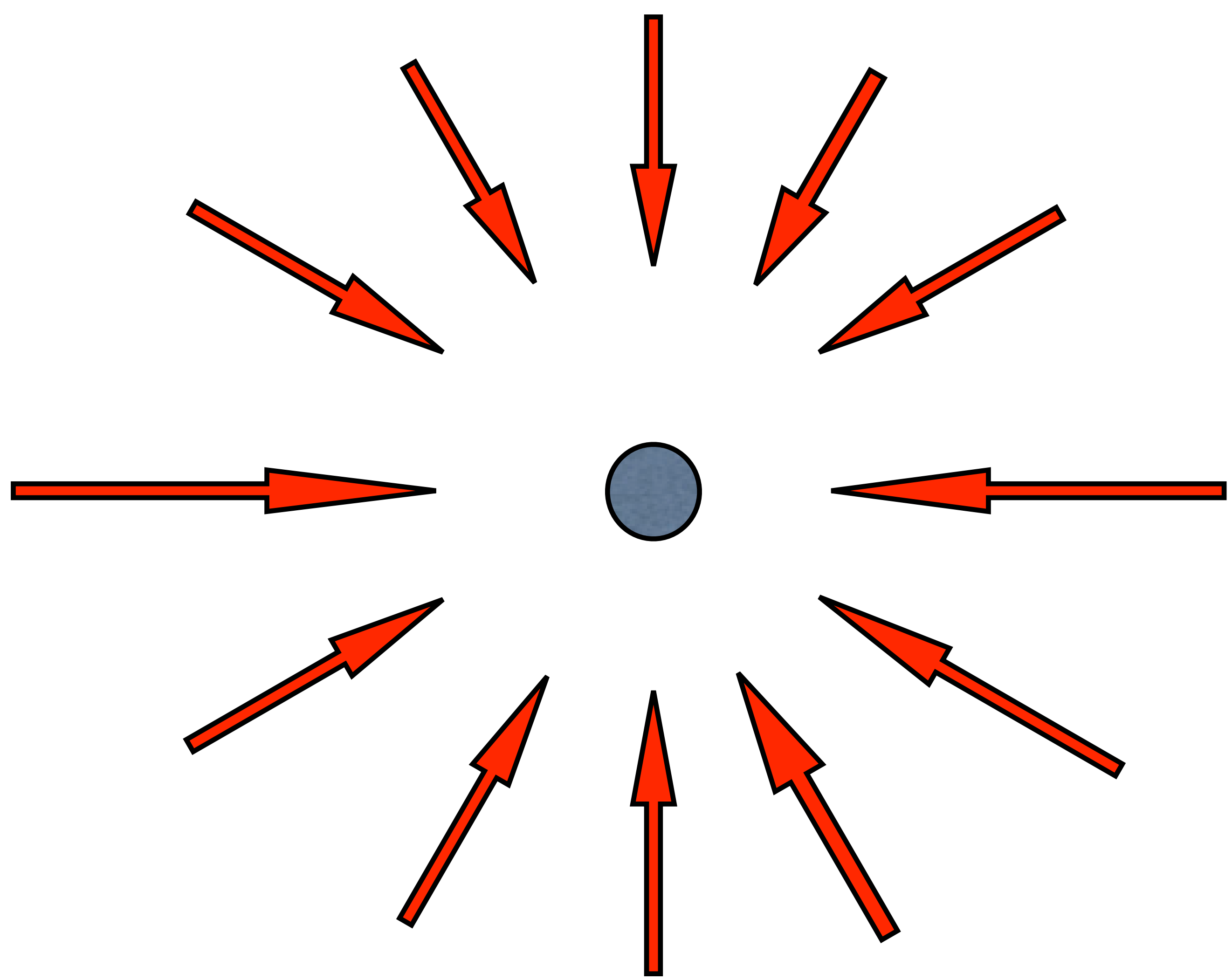




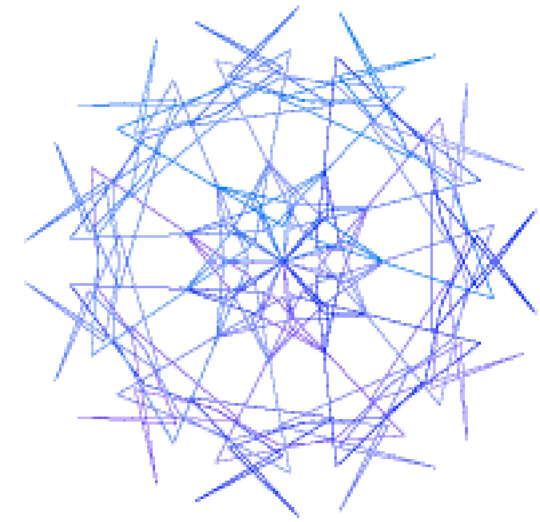
$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

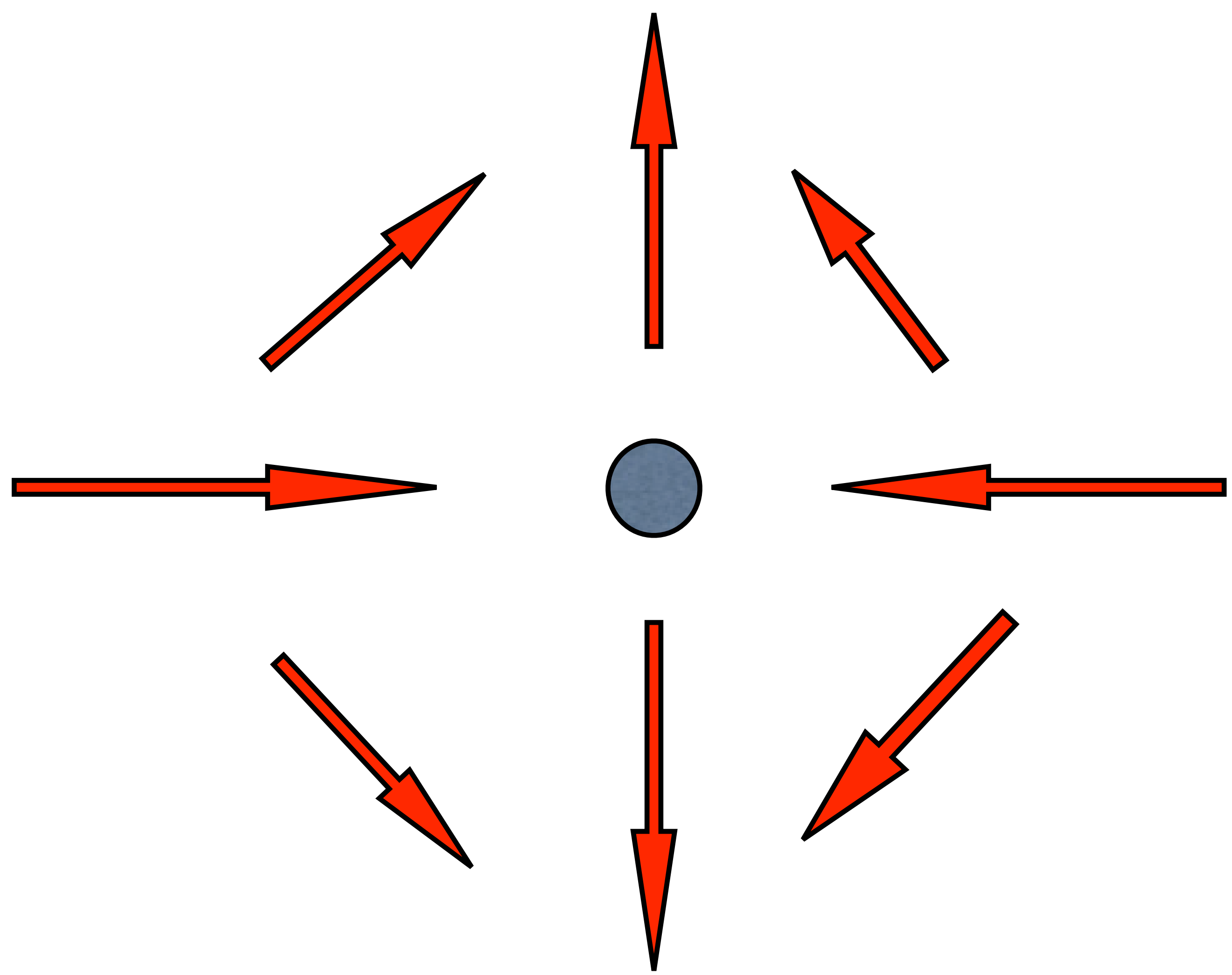
$$A\vec{x} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \vec{x}$$



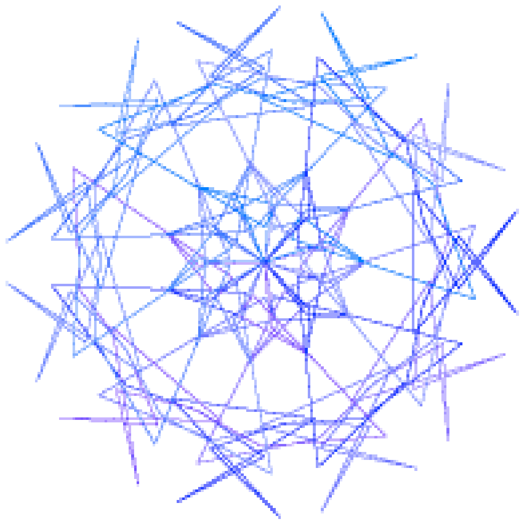


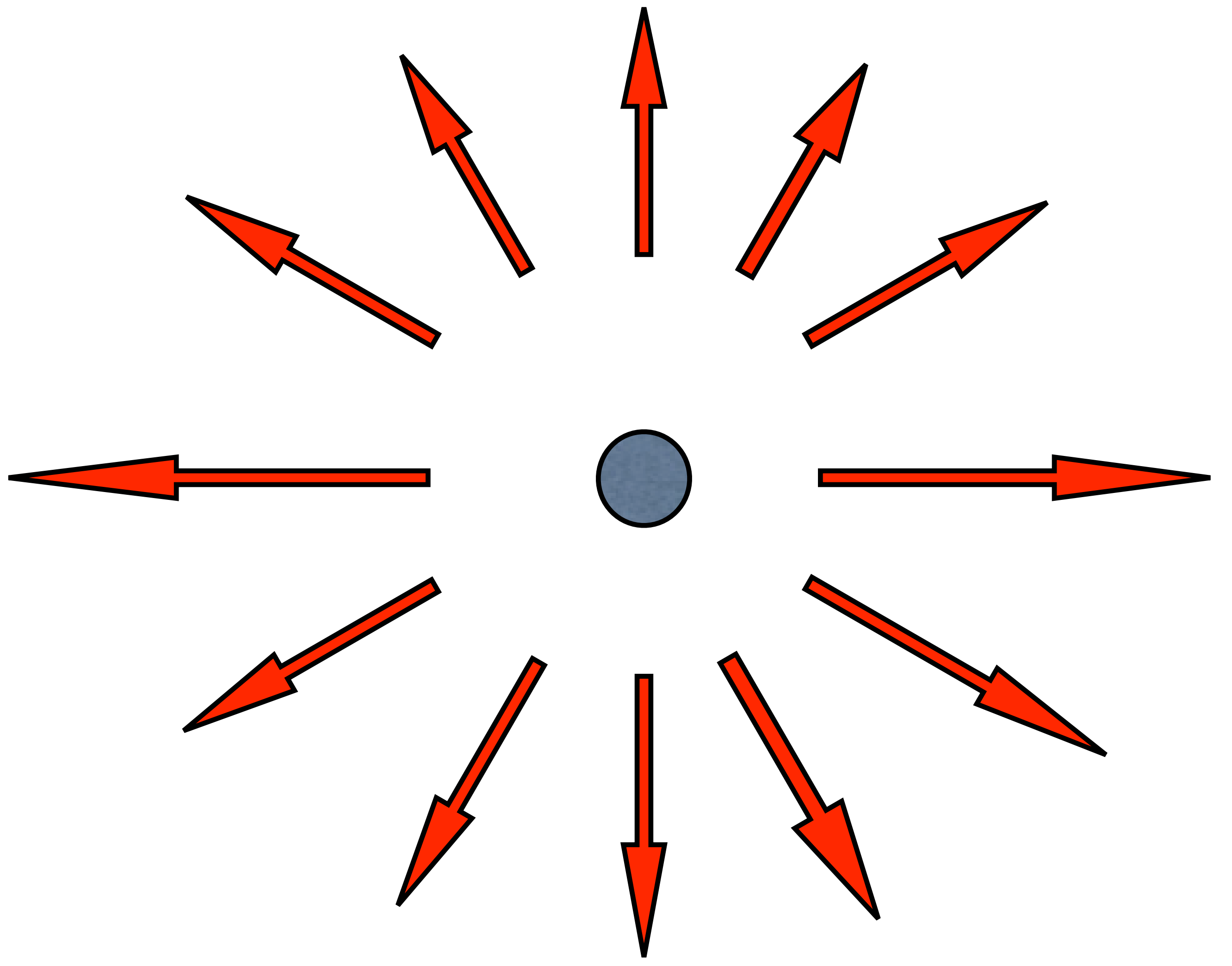
$$A\vec{x} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}$$



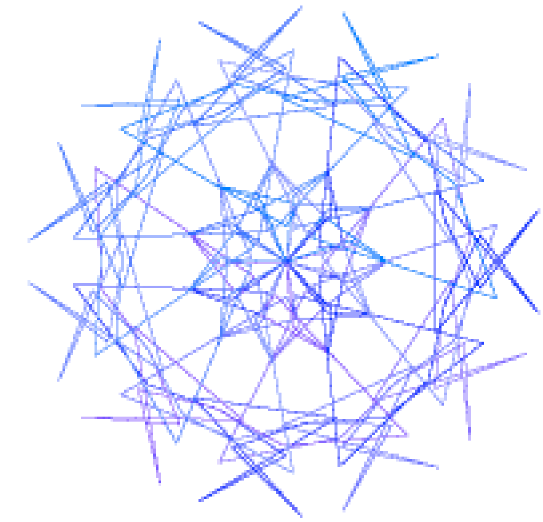


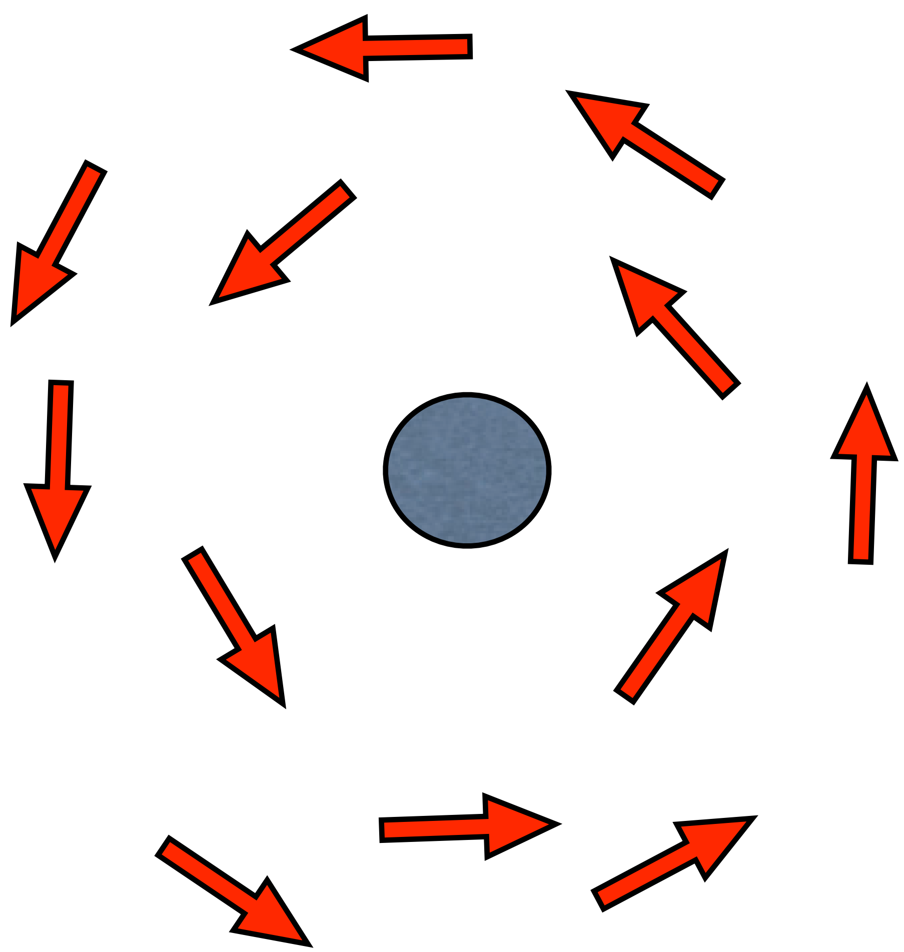
$$A\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x}$$



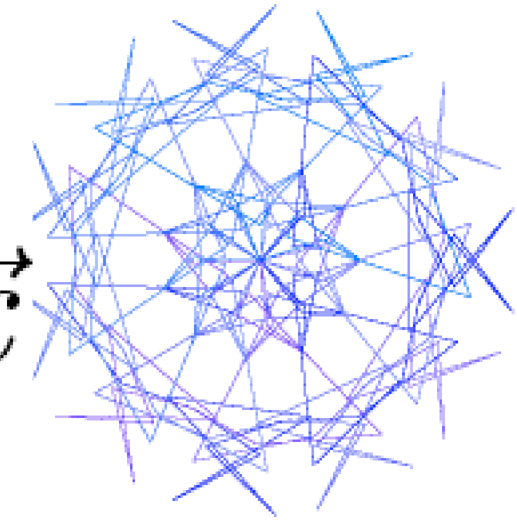


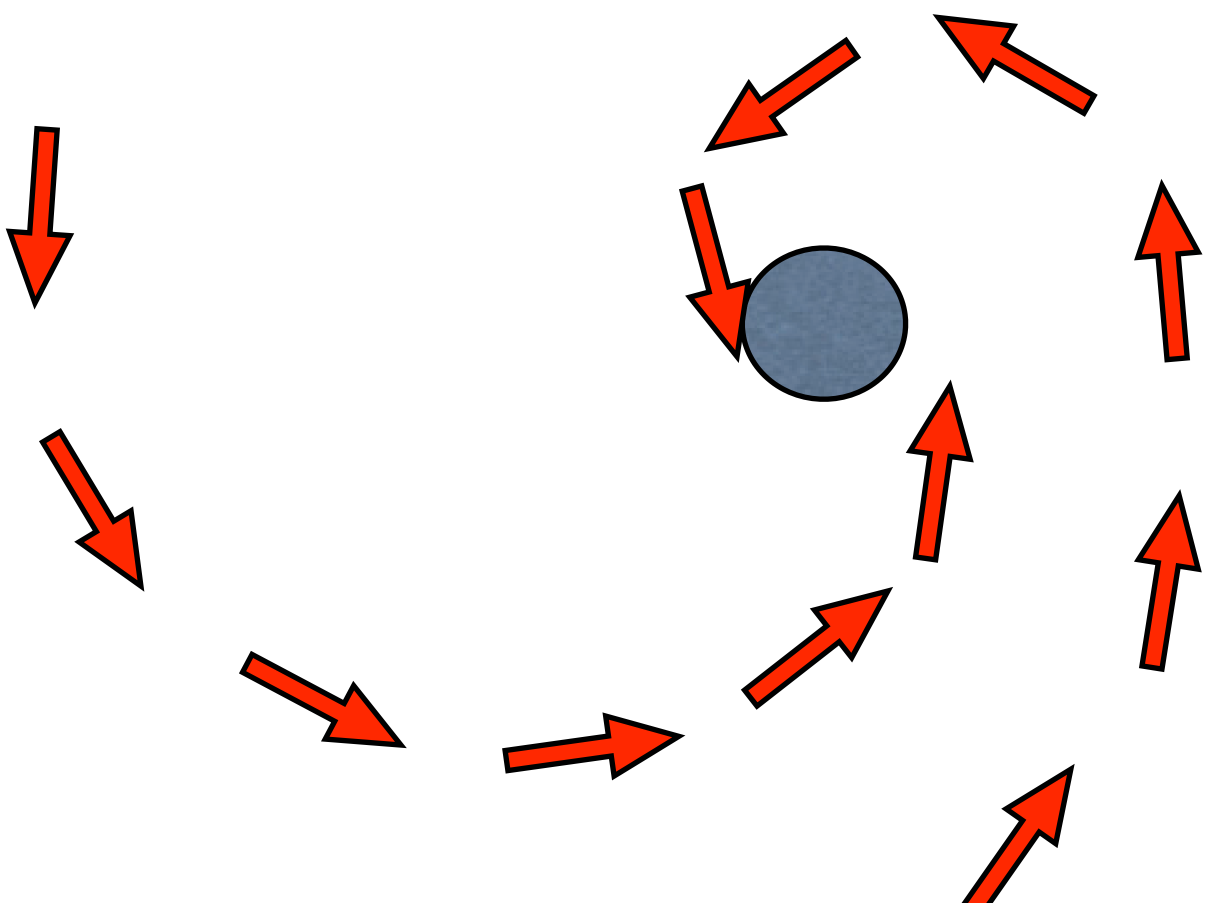
$$A\vec{x} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \vec{x}$$



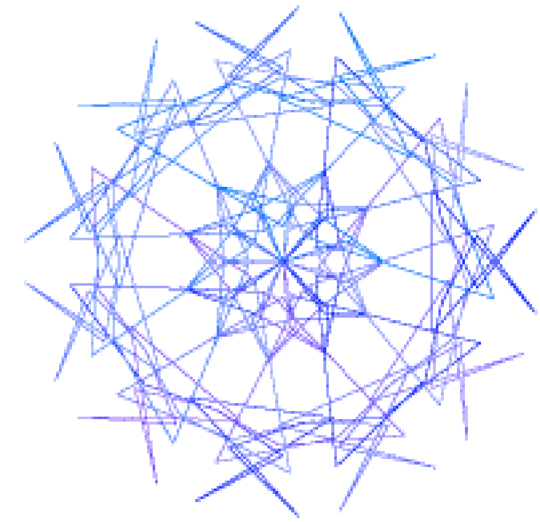


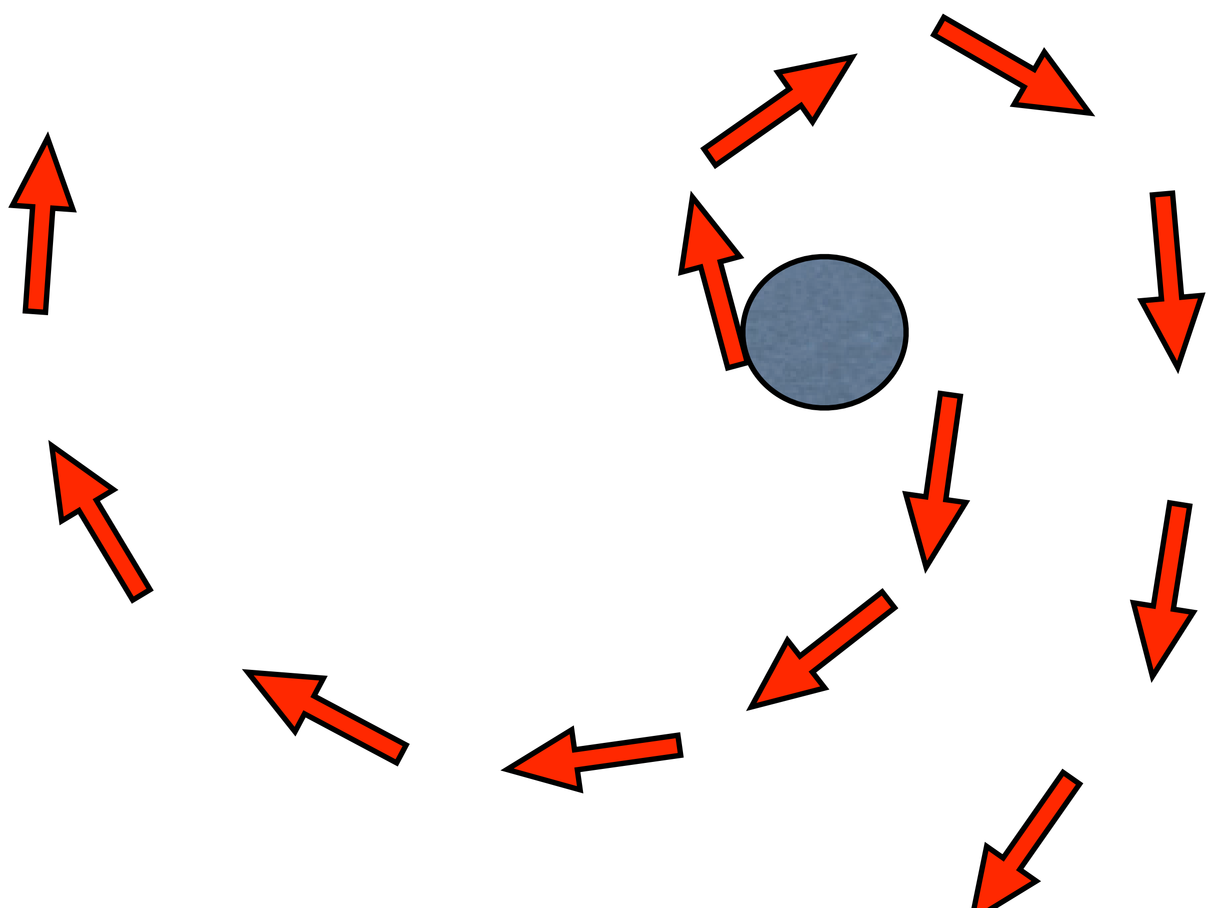
$$A\vec{x} = \frac{1}{2}\begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}\vec{x}$$





$$A\vec{x} = 2\begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}\vec{x}$$





Eigenvalues

The eigenvalues of A determine the stabilty of the origin.

Asymptotic stability

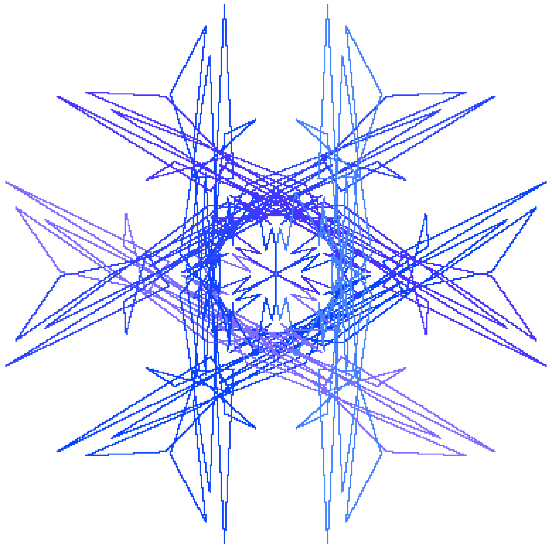
All eigenvalues have absolute value <1

Initial Value Problem

- Diagonalize A
- Write x as sum of eigenvector
- Write down solution

Example Blackboard

The Problem:



$$x(t+1) = x(t) + y(t)$$

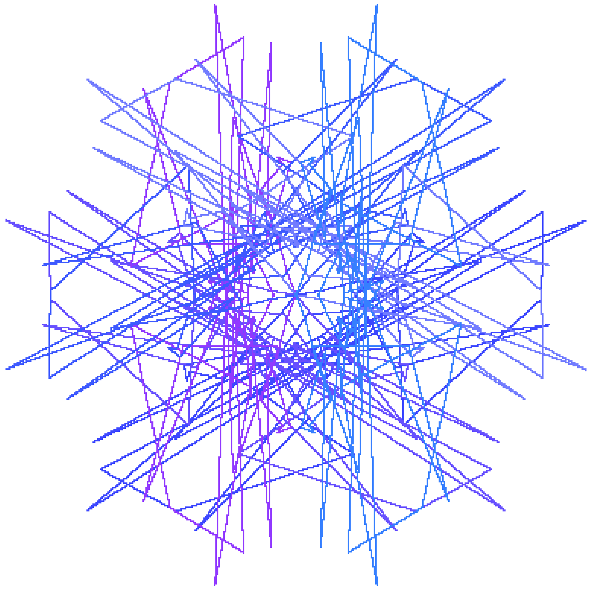
$$y(t+1) = x(t) - y(t)$$

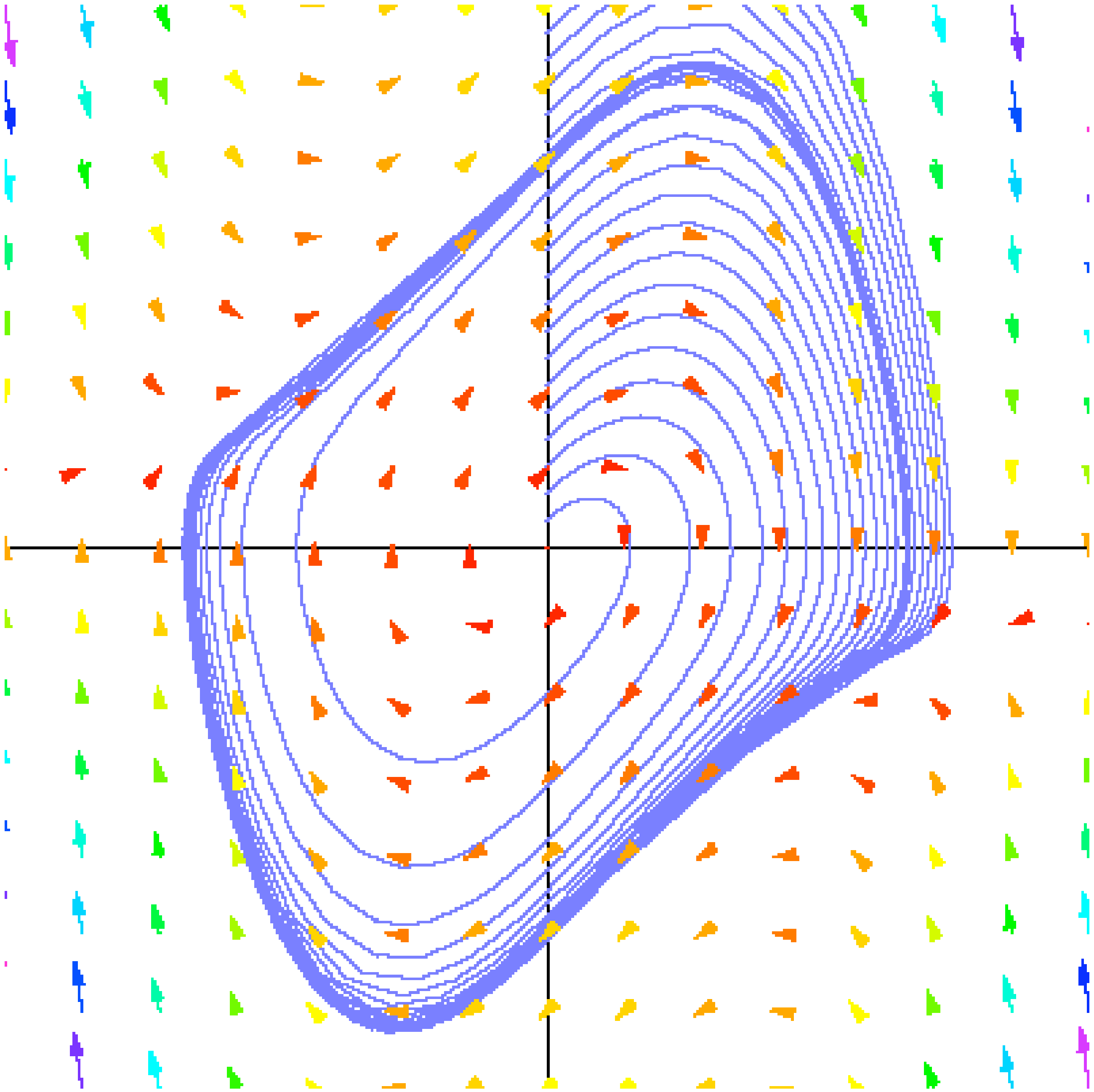
x	y
2	2
4	0
4	4
...	...

III) Differential Equations

- Solving the system
- Analyze the phase space
- Determine stability

Differential equations





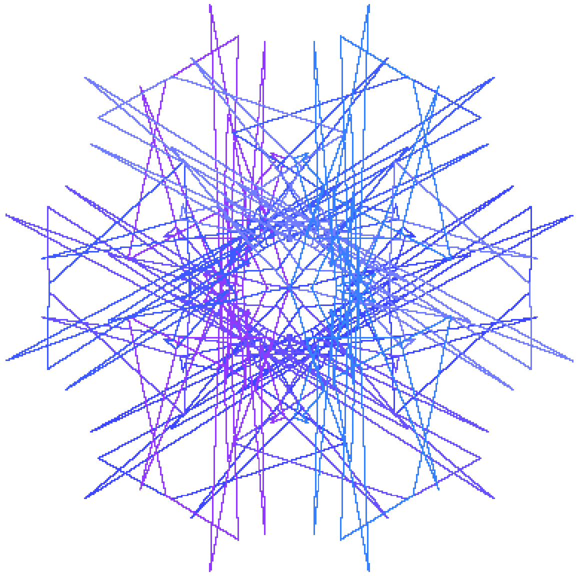
$$\dot{x}=f(x,y)$$

$$\dot{y}=g(x,y)$$

We covered a lot of material!

But we like it extreme!

Linear Differential equations in the plane



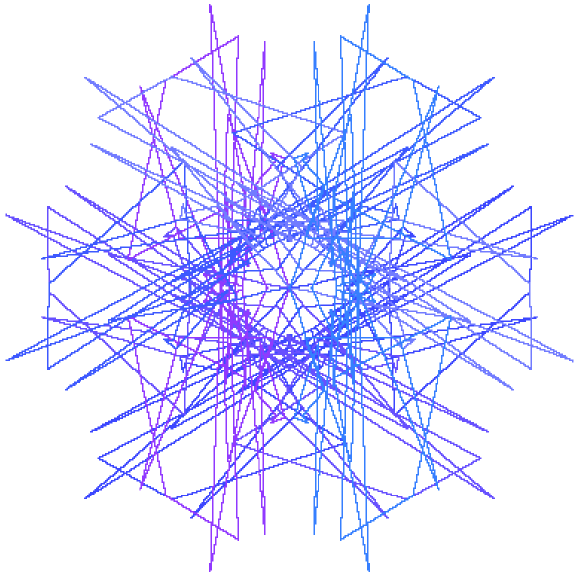
$\dot{x} = ax + by$

$\dot{y} = cx + dy$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The eigenvalues of A determine the dynamical behavior.

One dimensional case



$$\frac{d}{dt}x = \lambda x$$

$$x(t) = e^{\lambda t}x(0)$$

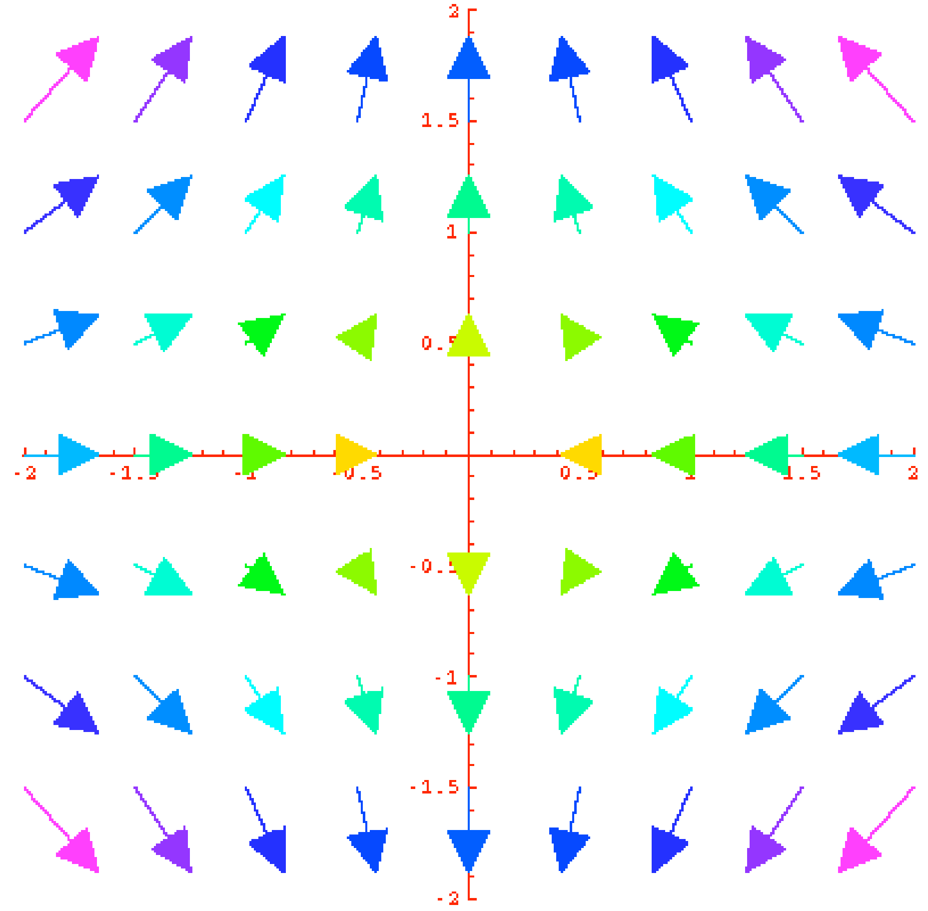
The mother of all linear differential equations

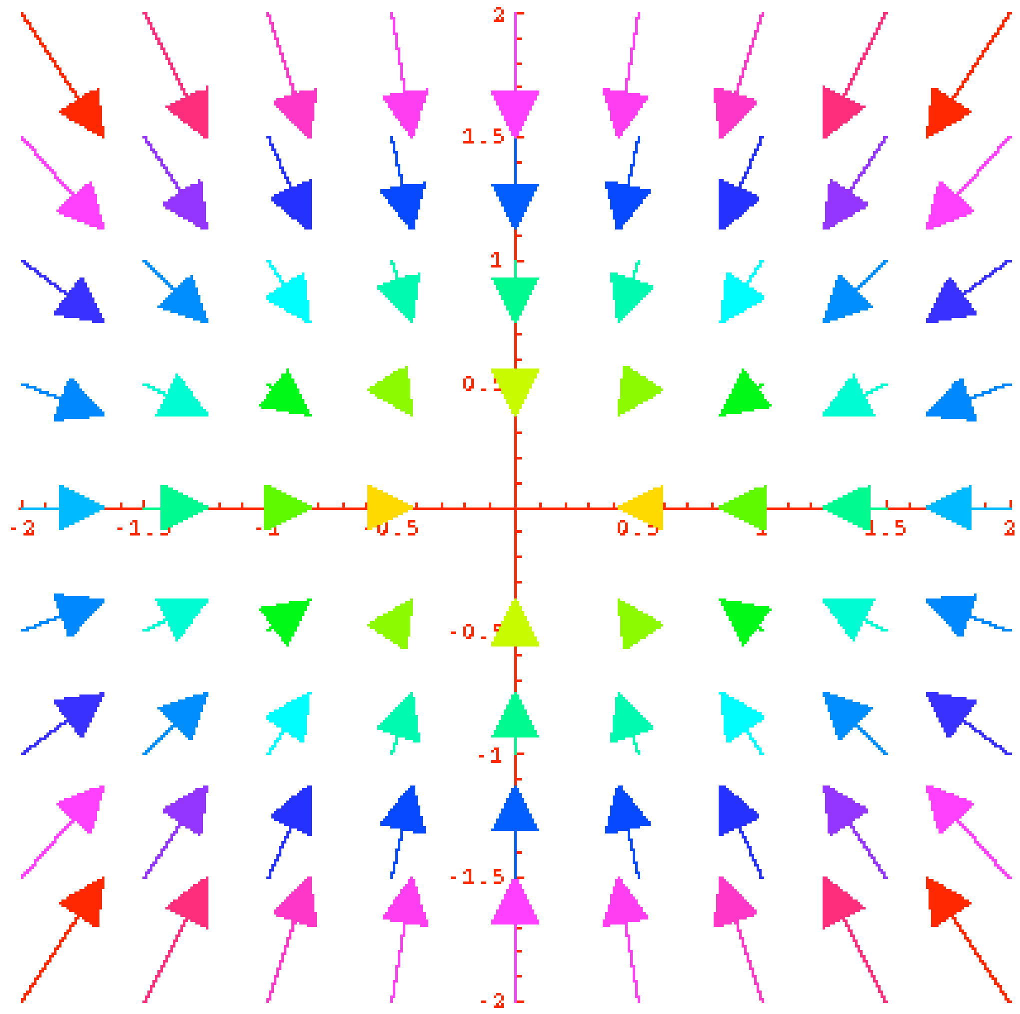
Real, nonzero eigenvalues

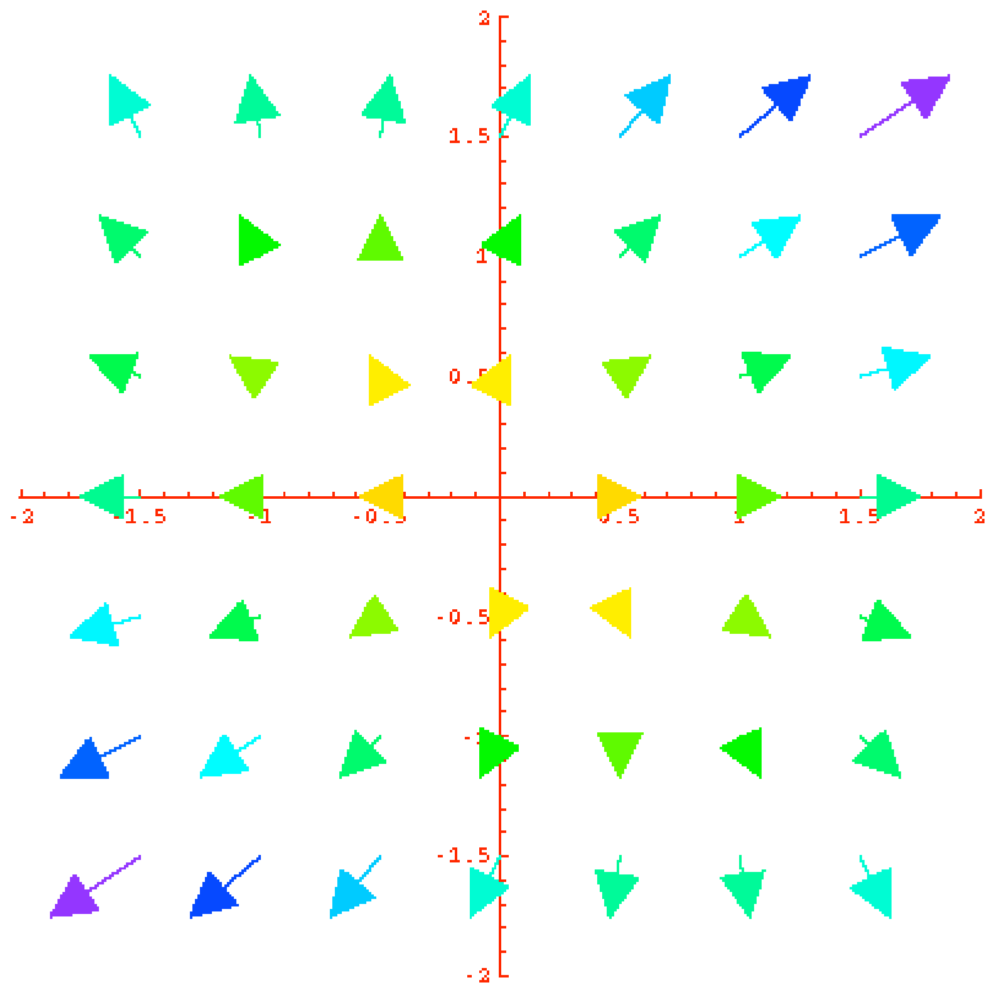
$\dot{x} = ax + by$

$\dot{y} = cx + dy$

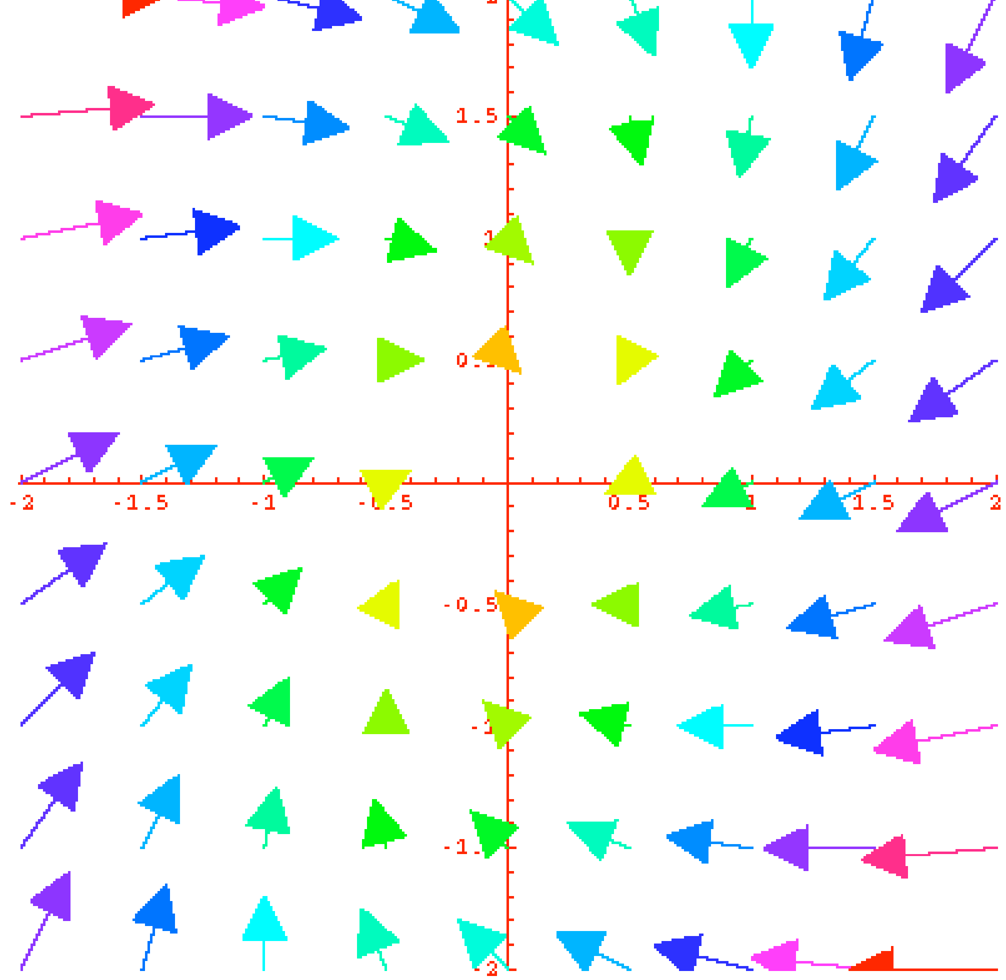
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

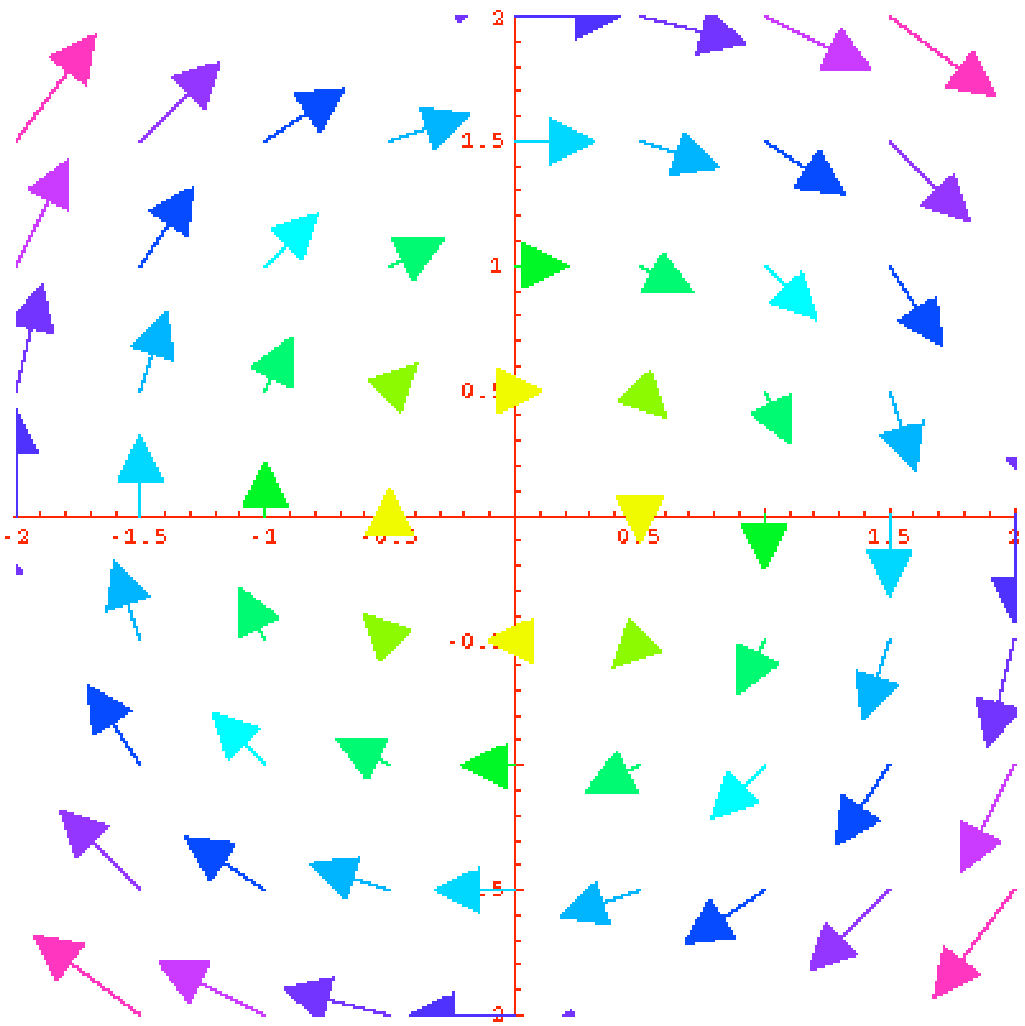


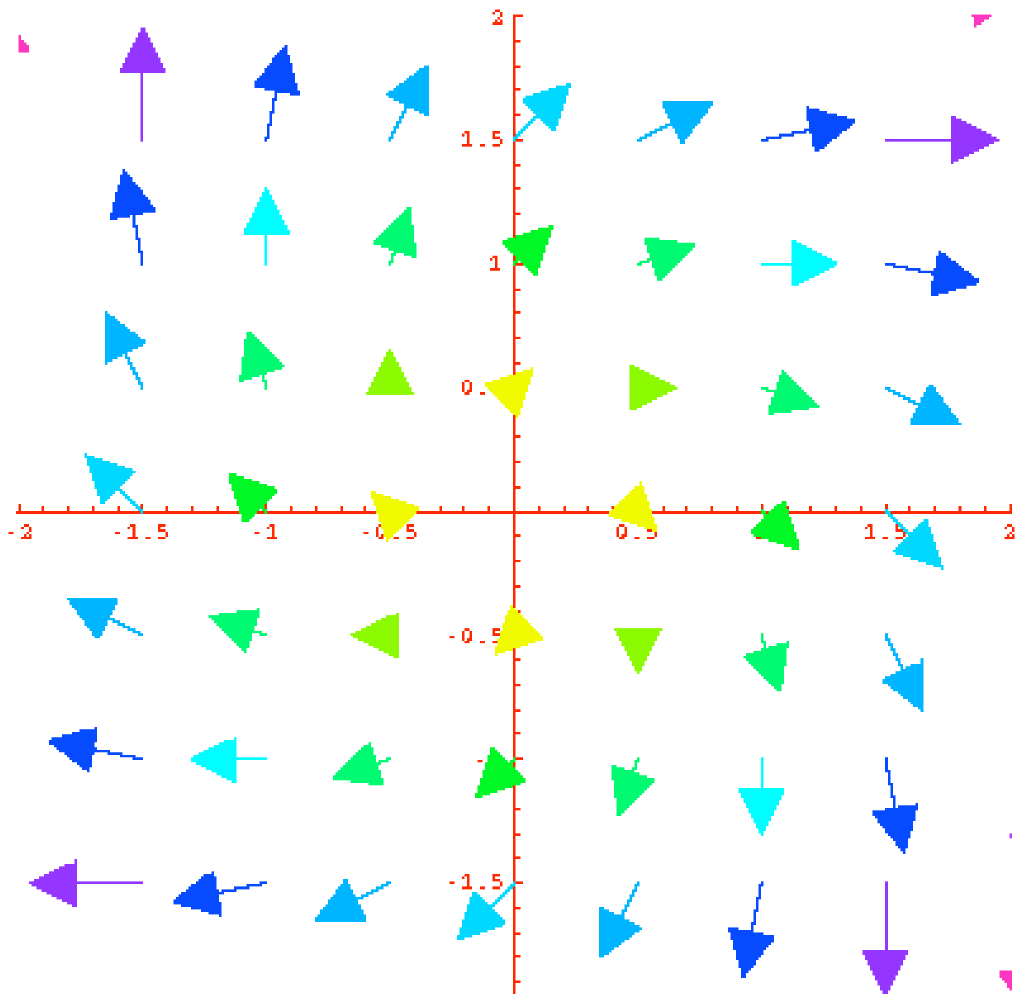




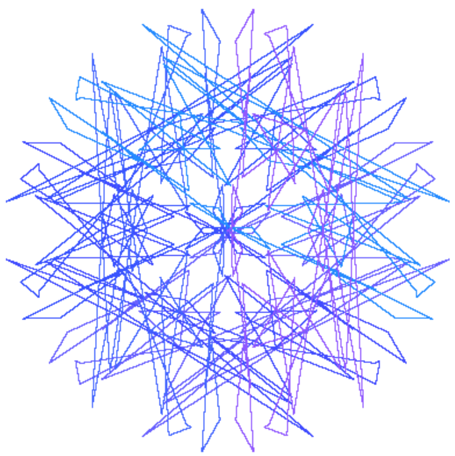
Complex eigenvalues

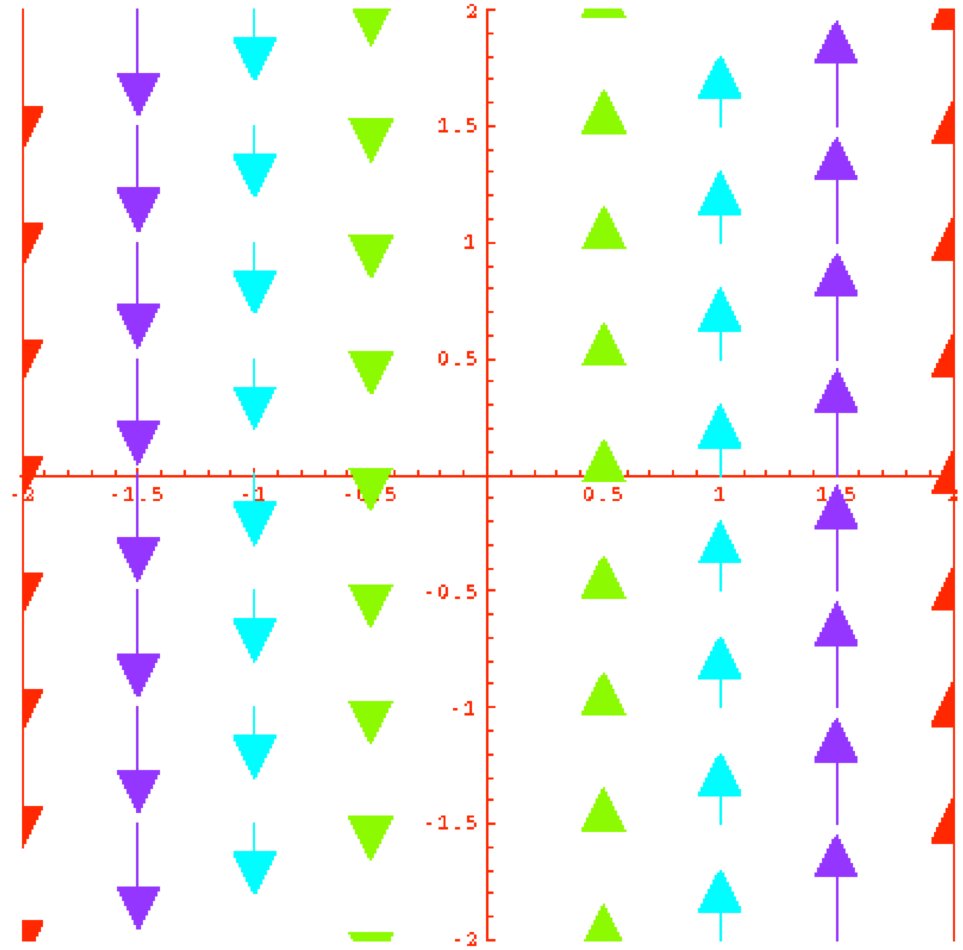


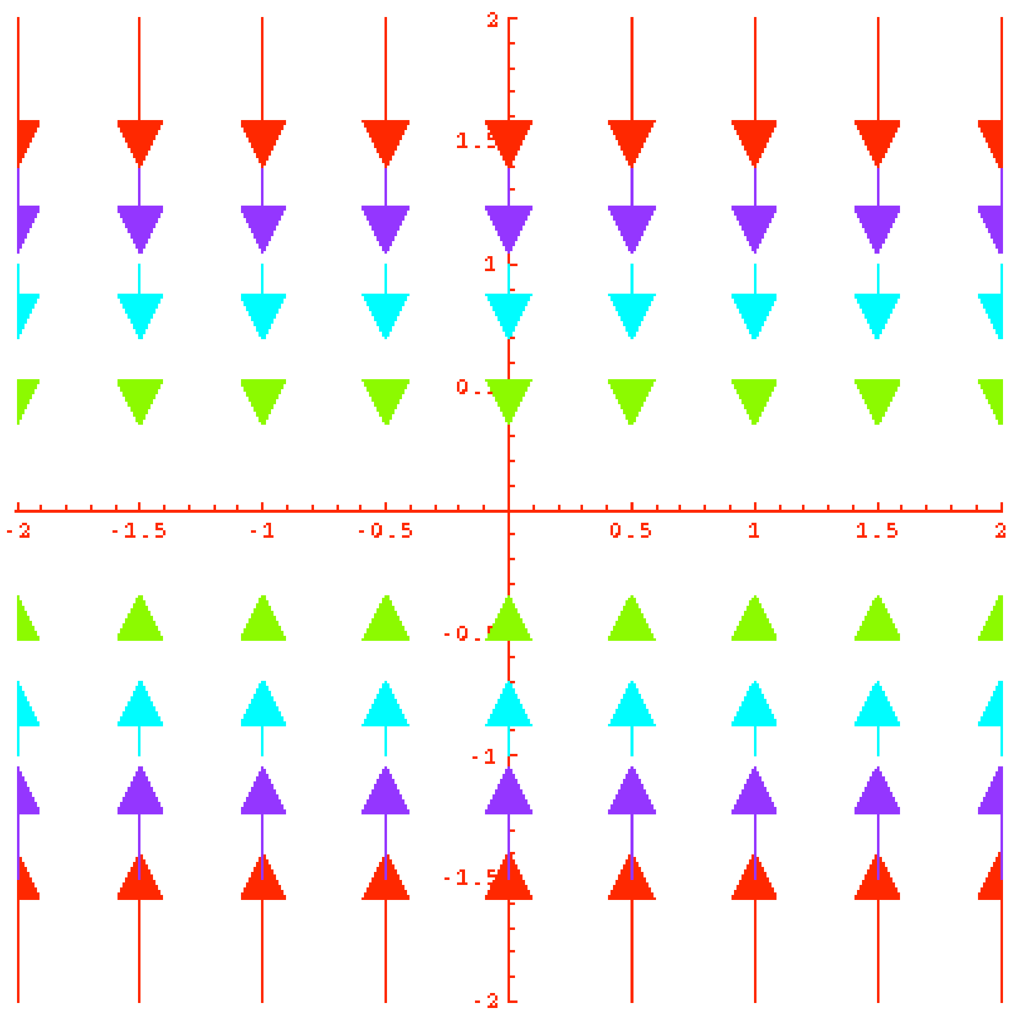


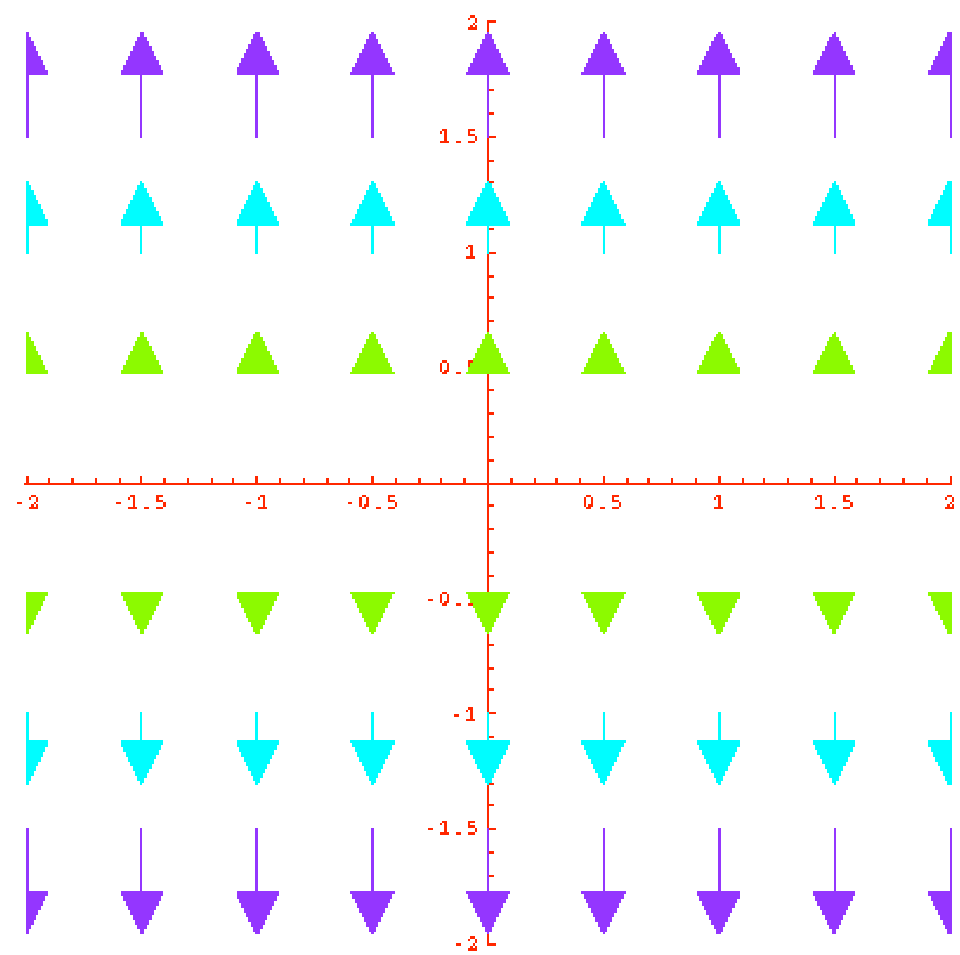


Some zero eigenvalue

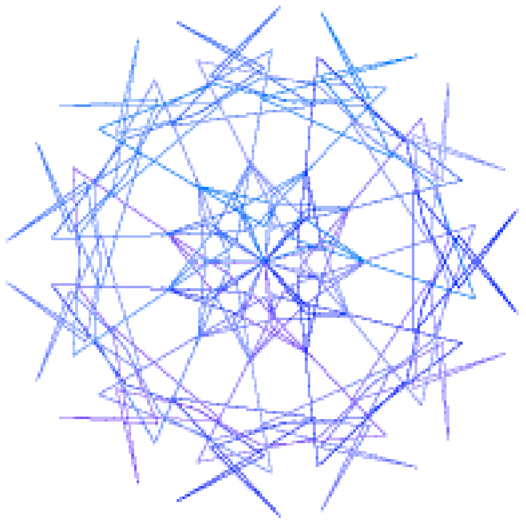


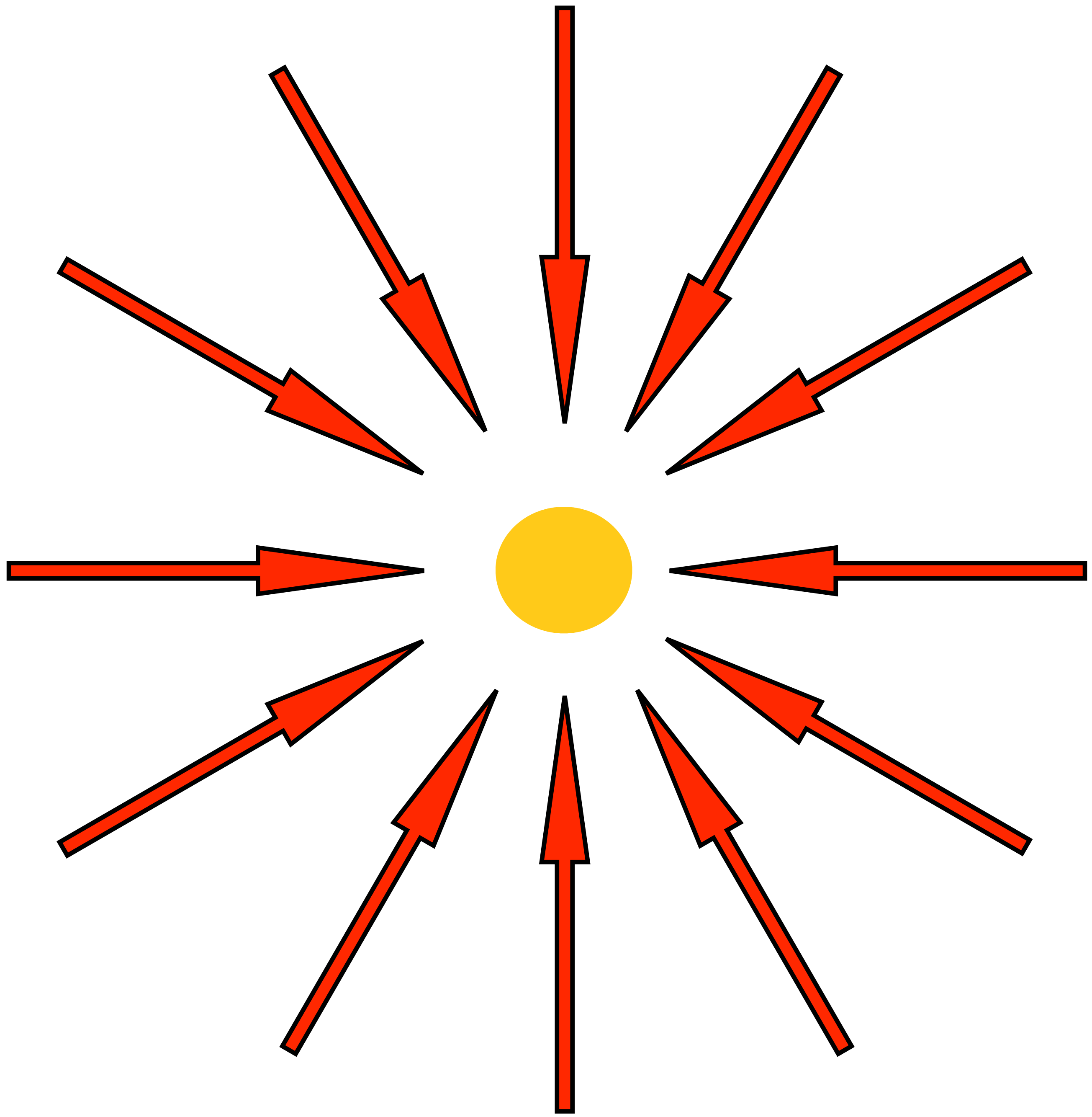




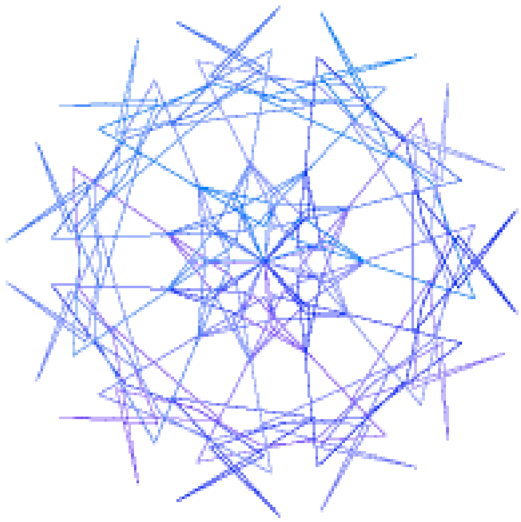


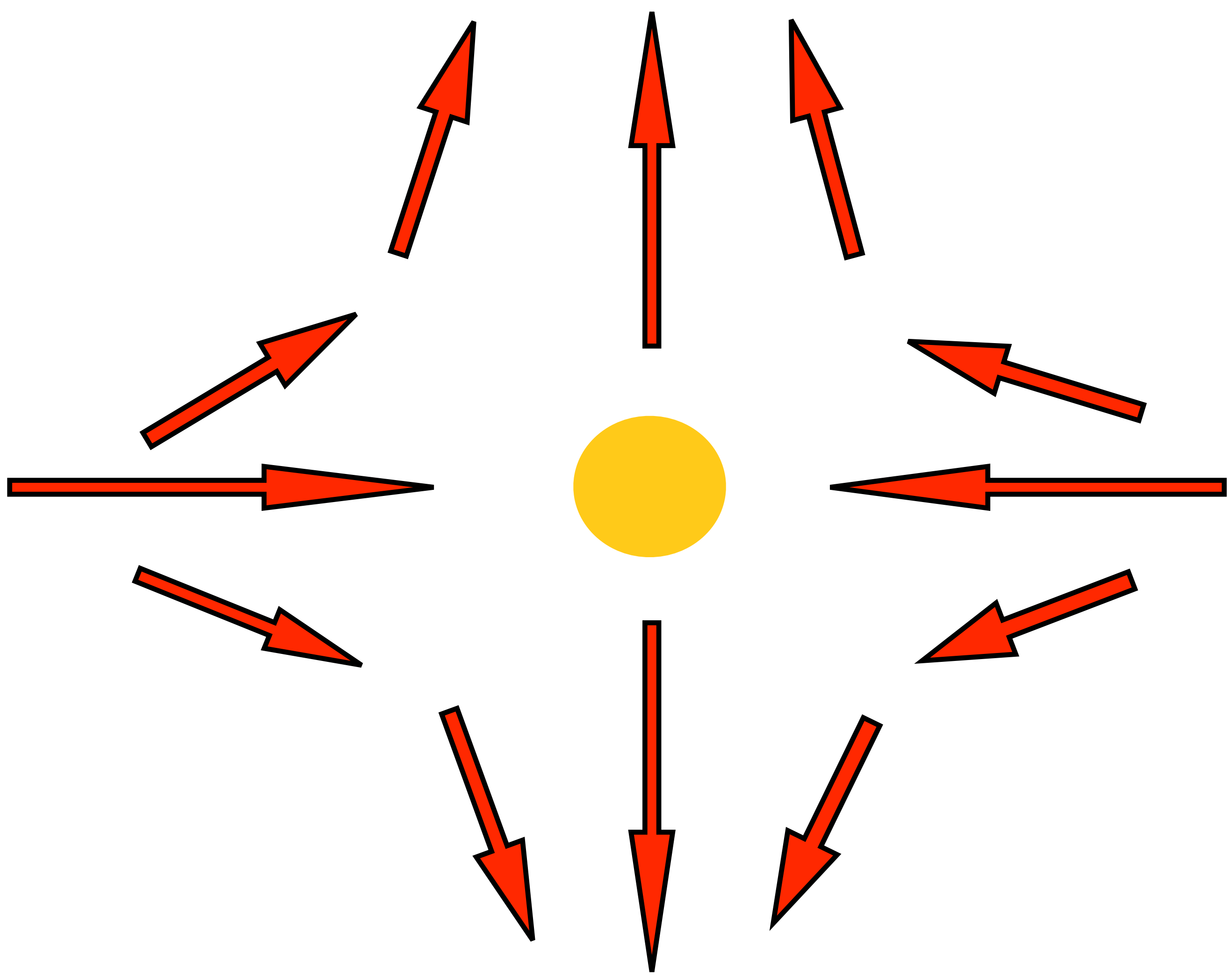
$$\frac{d}{dt}\vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \vec{x}$$



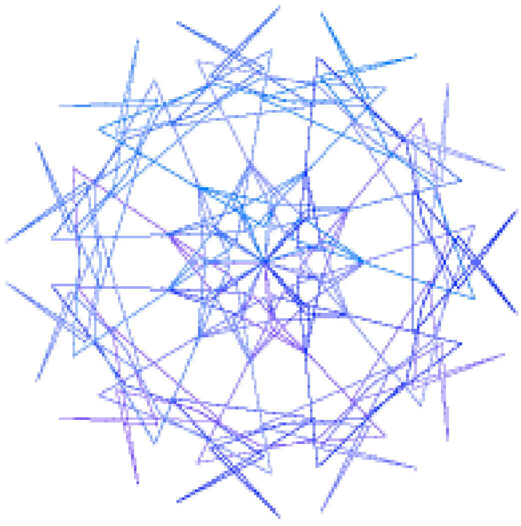


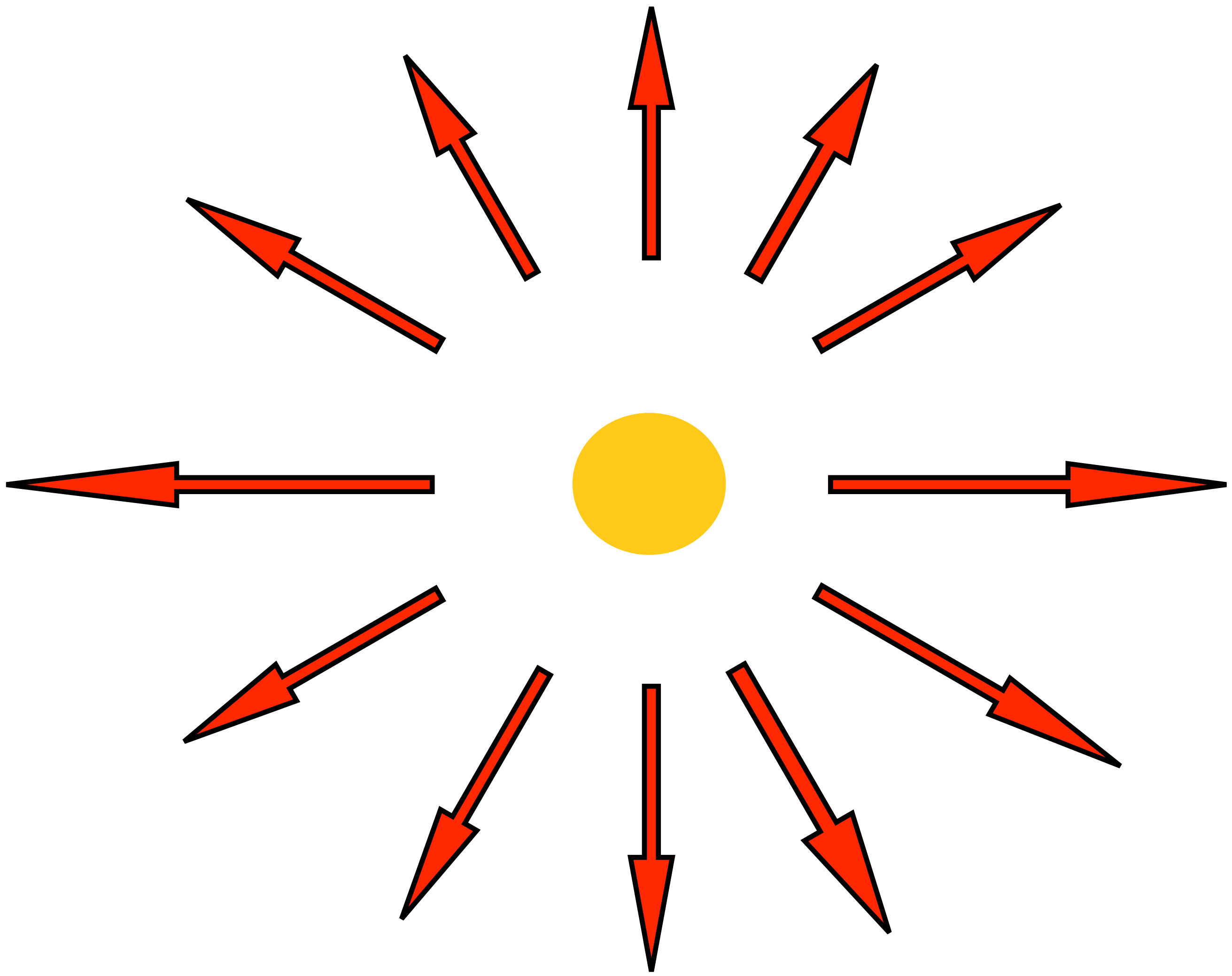
$$\frac{d}{dt}\vec{x} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \vec{x}$$



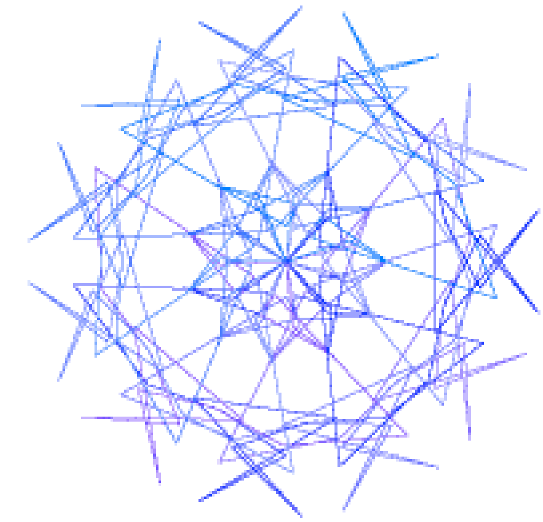


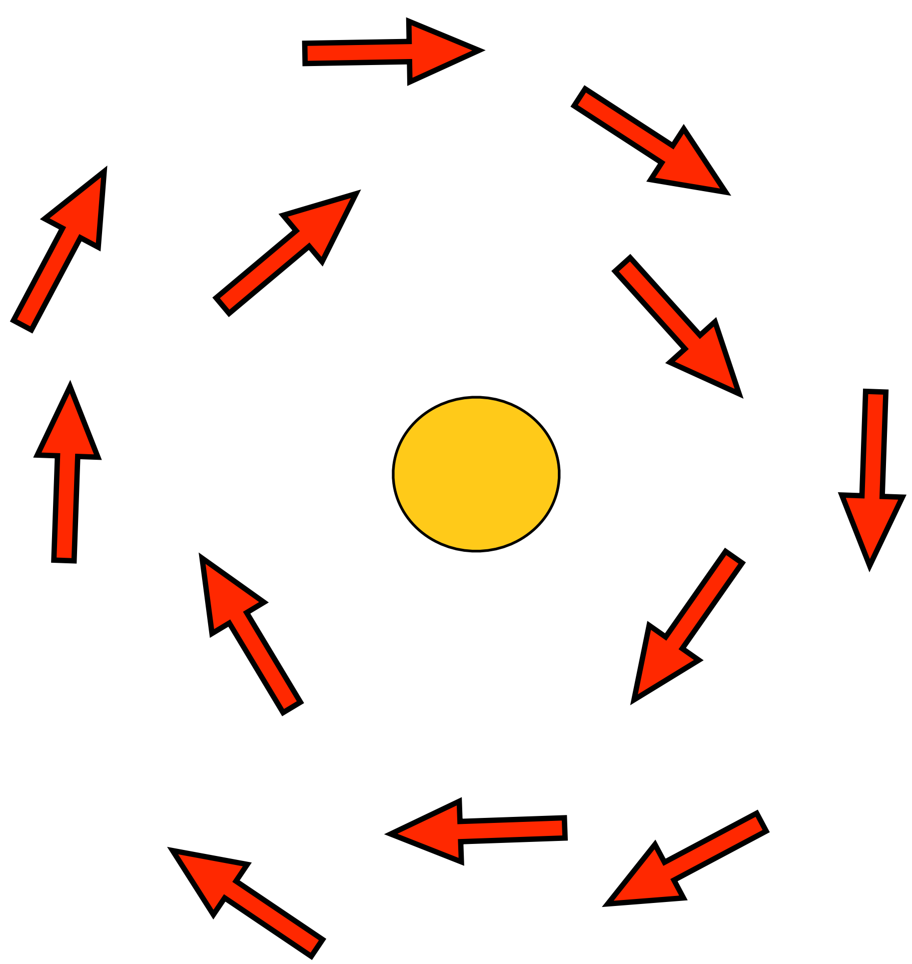
$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \vec{x}$$



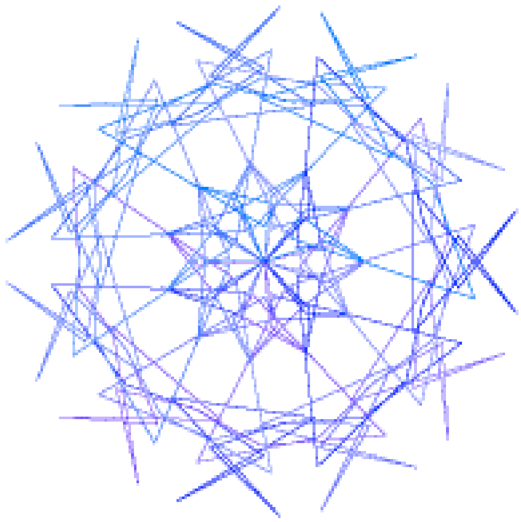


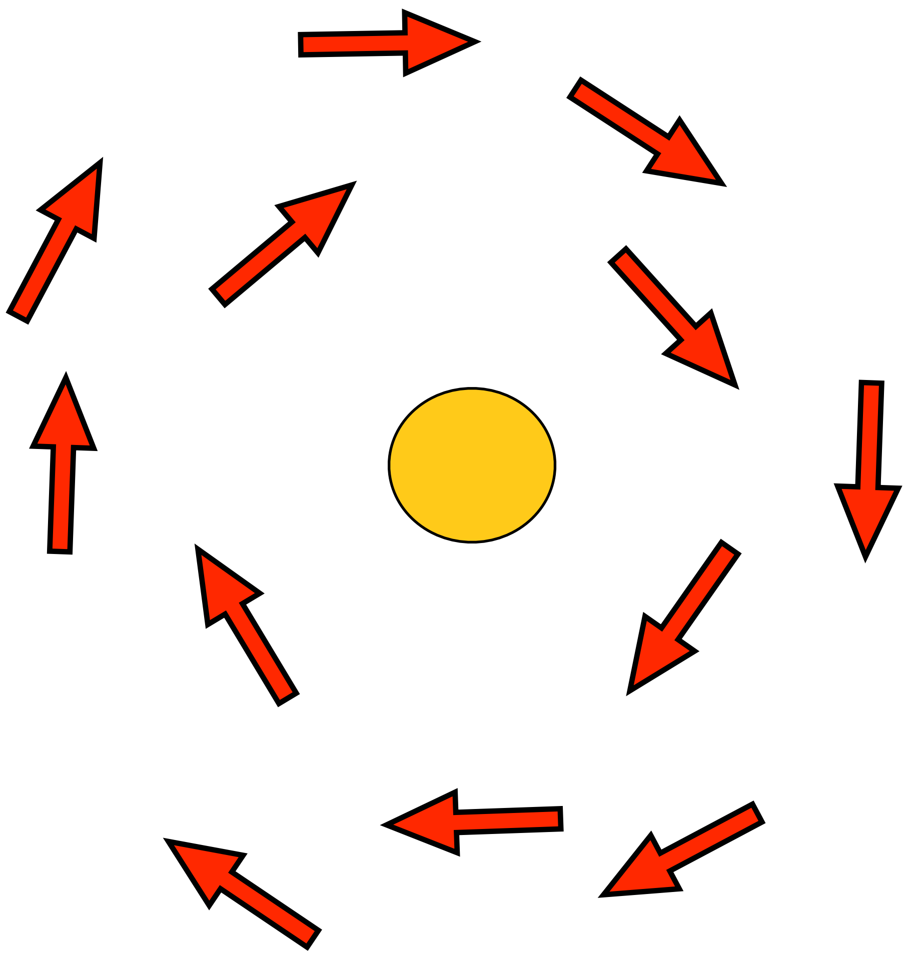
$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$$

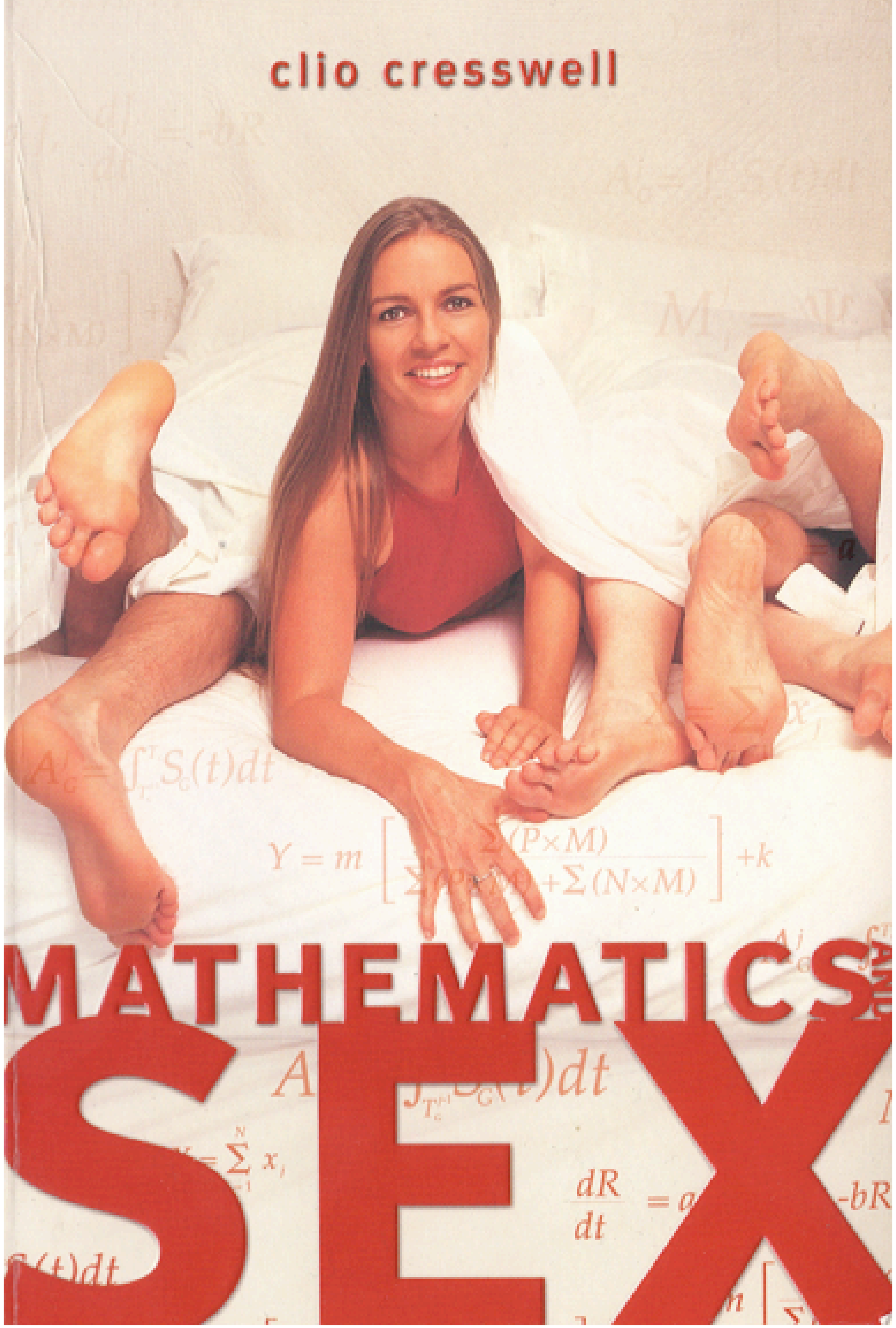




Do the eigenvalues determine the rotation direction?





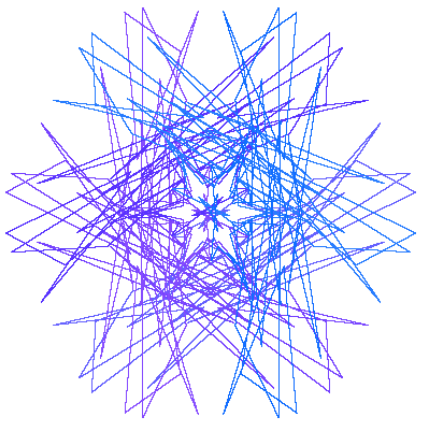


Mathematics and Sex

Clio Cresswell, 2003

Chapter 1

LOVE, SWEEEET LOVE



In the late '80s, a Harvard lecturer by the name of Steven Strogatz suggested an unusual class exercise to his students. The day's topic would be the Mathematics of Love. Professor Strogatz's motivations were plain cheeky. Confronted with the challenge of capturing his students' attention on the predictive powers of equations, he reworded a common undergraduate mathematics problem into a language he thought the students would relate to: the evolution of the love affair between Romeo and Juliet. His ingenuity should not be taken lightly: turning a group of hormone-raging twenty-year olds into utterly focused mathematical geniuses is a complex task. I wish I had been in his class to witness the full event.

Steven Strogatz didn't base his class exercise on extensive psychological research; he was just a Harvard lecturer having a bit of fun. But little did he realise he was actually beginning to

make some mathematical sense of one of the great human emotions.

He presented the problem like this:

> Romeo is in love with Juliet, but in our version of the story, Juliet is a fickle lover. The more Romeo loves her, the more Juliet wants to run away and hide. But when Romeo gets discouraged and backs off, Juliet begins to find him strangely attractive. Romeo, on the other hand, tends to echo her: he warms up when she loves him, and grows cold when she hates him.

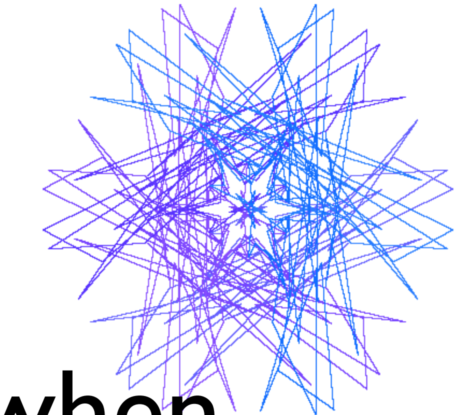
As you can see, emotions are a bit all over the place in this relationship. The question is, will they ever settle? What kind of relationship can Romeo and Juliet look forward to? The point of the exercise is to show how equations give insight into these real-life dilemmas. And no doubt many of the students related to the example.

The first step towards mathematical insight is to rewrite the terms of Romeo and Juliet's fickle affair mathematically. The translation is:

[redacted]

where R is for Romeo, and J for Juliet. How the letters are combined mimics how Romeo and Juliet find themselves interacting. For mathematicians, translating the problem into equations like this is natural. Mathematics is the study of patterns and this problem simply concerns behavioural patterns. Behavioural patterns are not static though and that's an important characteristic to

Romeo and Juliet



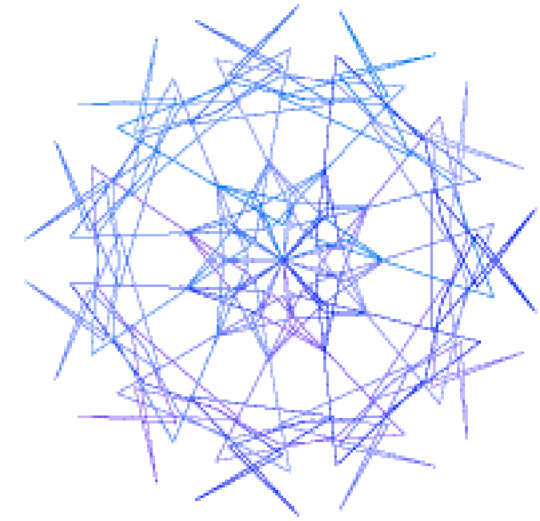
Romeo warms up when given more love,
Juliet wants to run away when being desired.

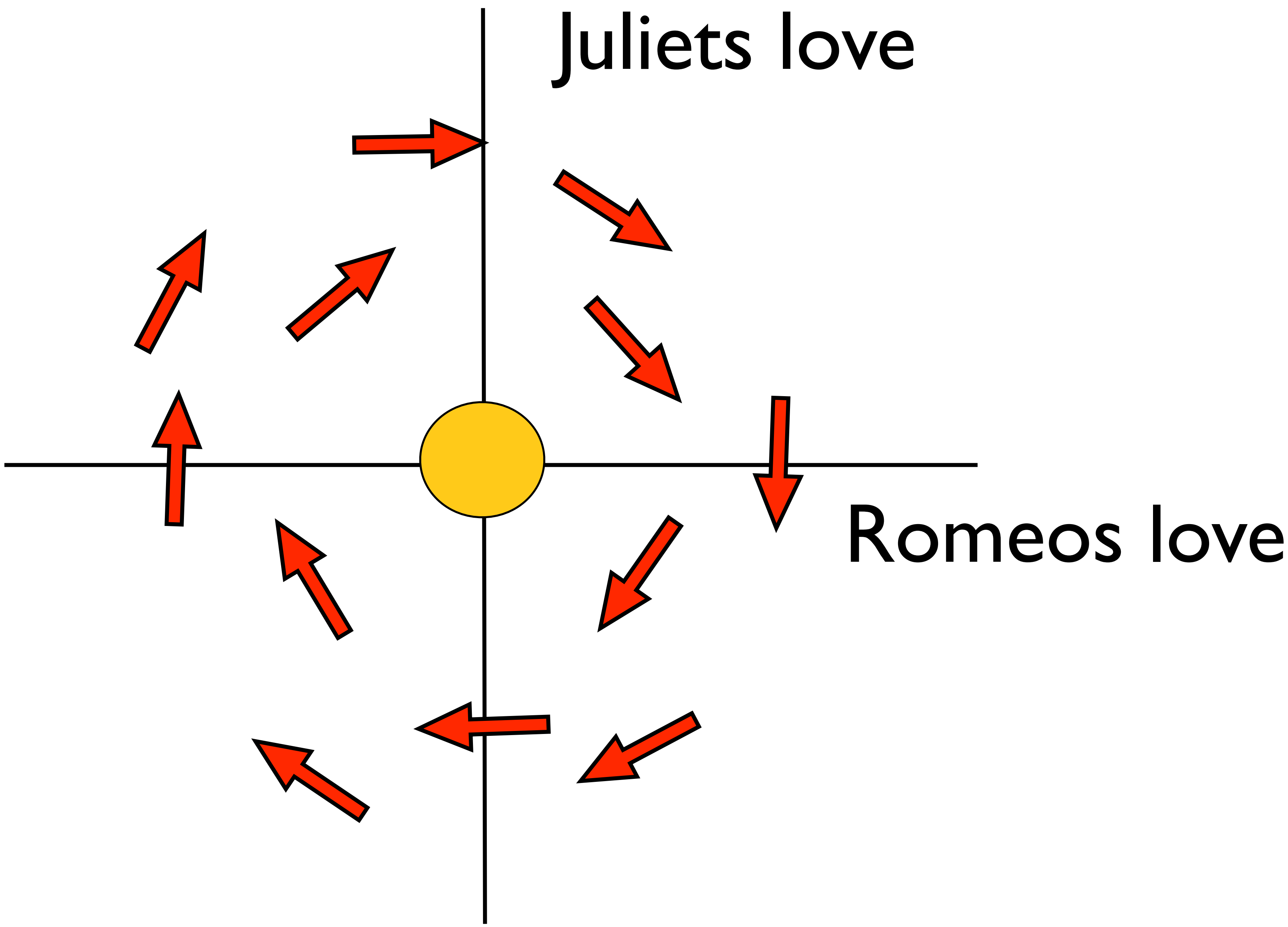
Problem: analyze, what happens with this love relationship!

$$A = \begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix}$$

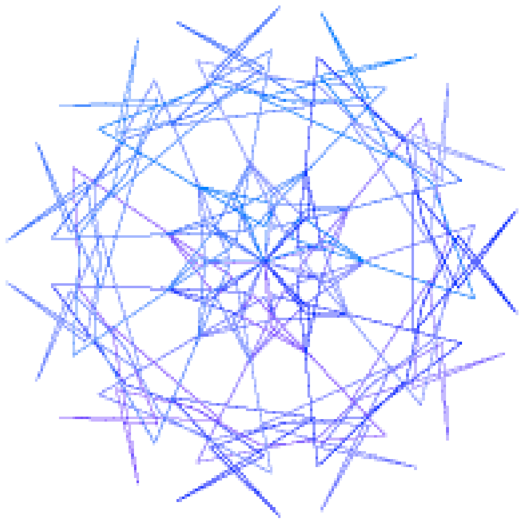
$$\dot{x} = a\,y$$
$$\dot{y} = -b\,x$$

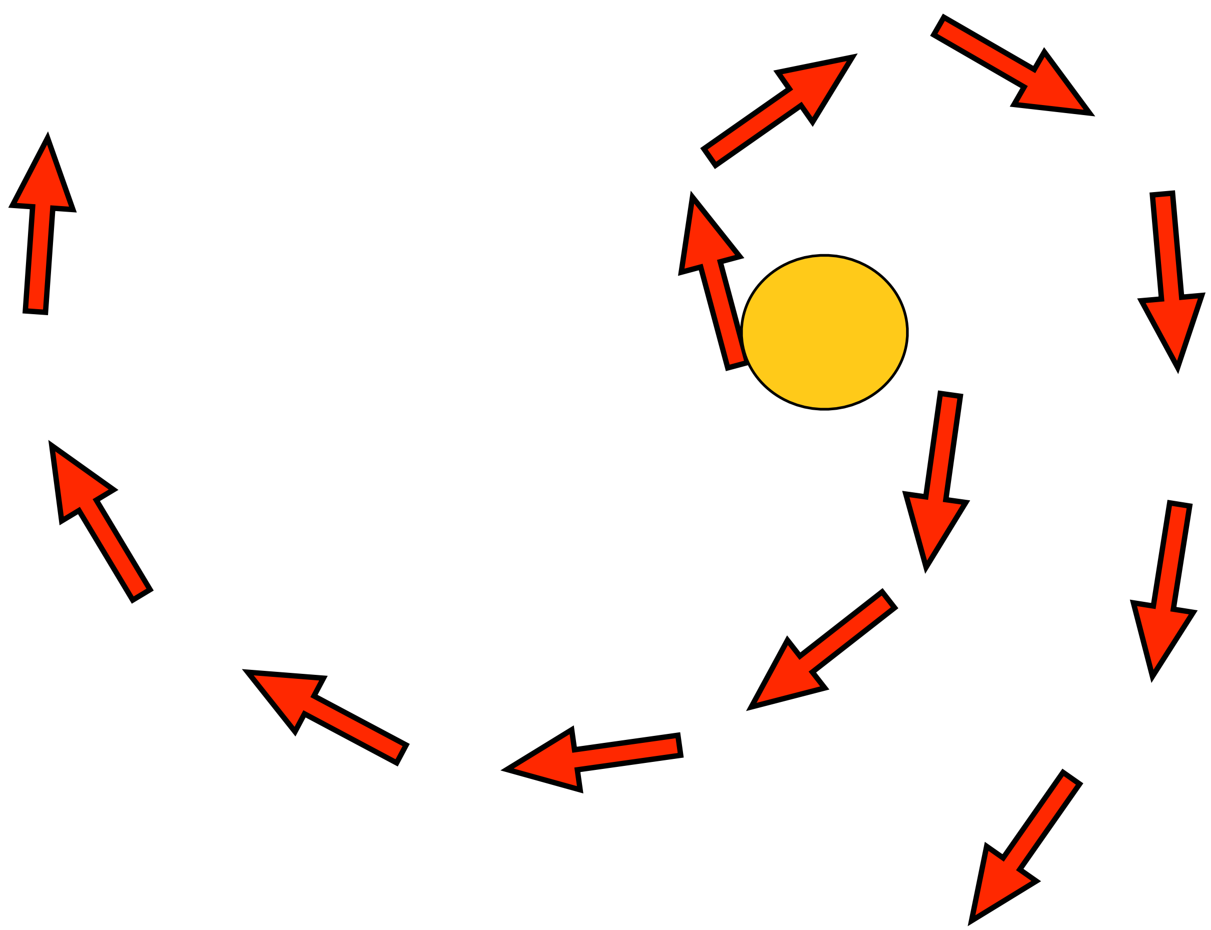
$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$$



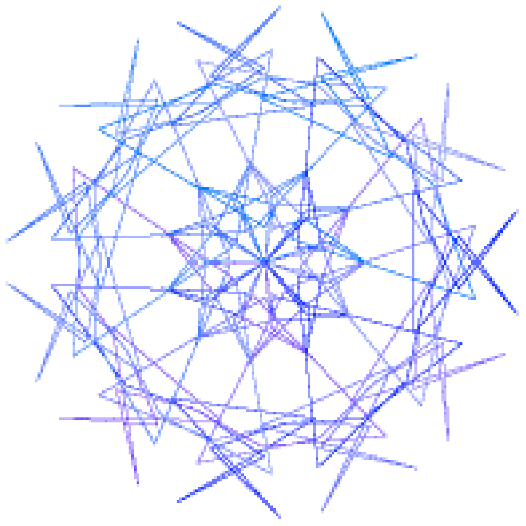


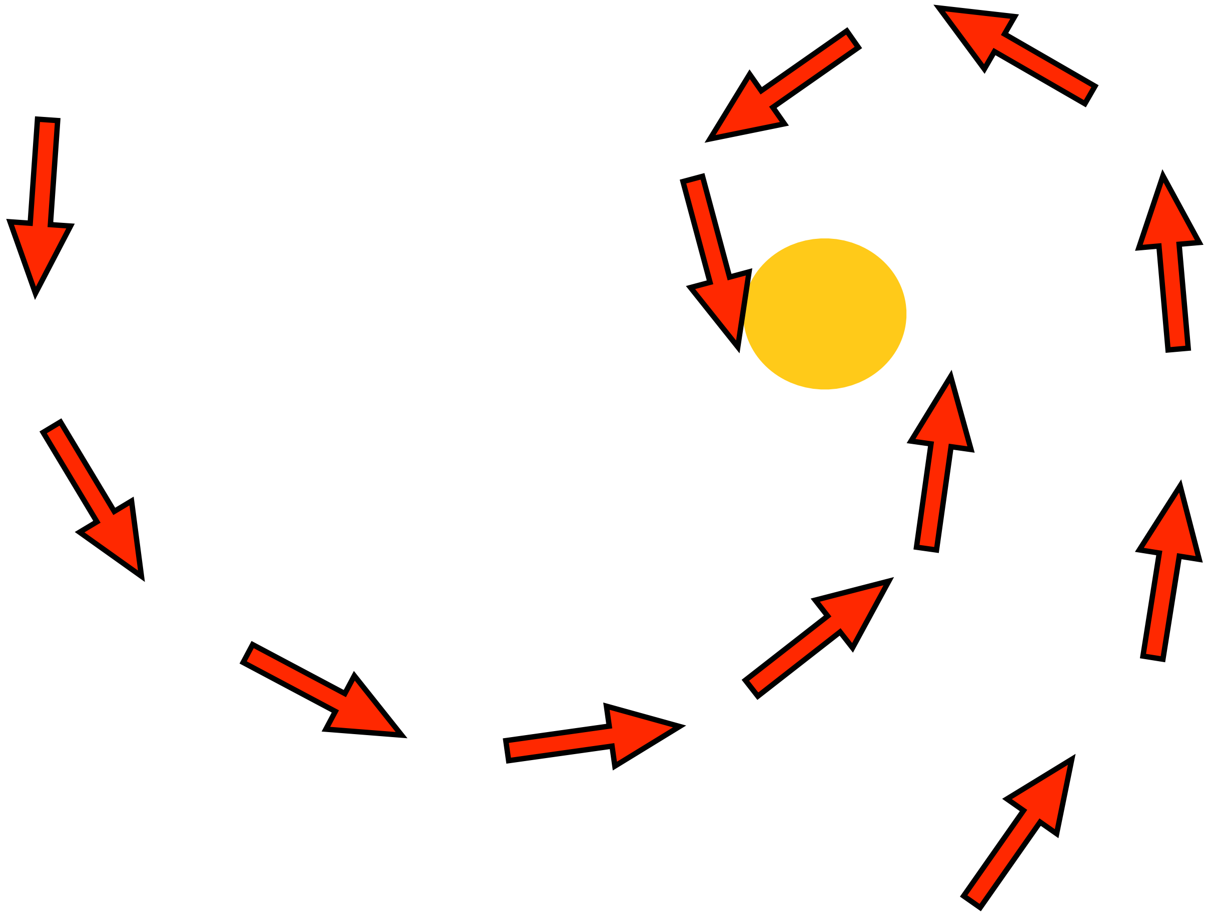
$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}$$





$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \vec{x}$$





Stability examples

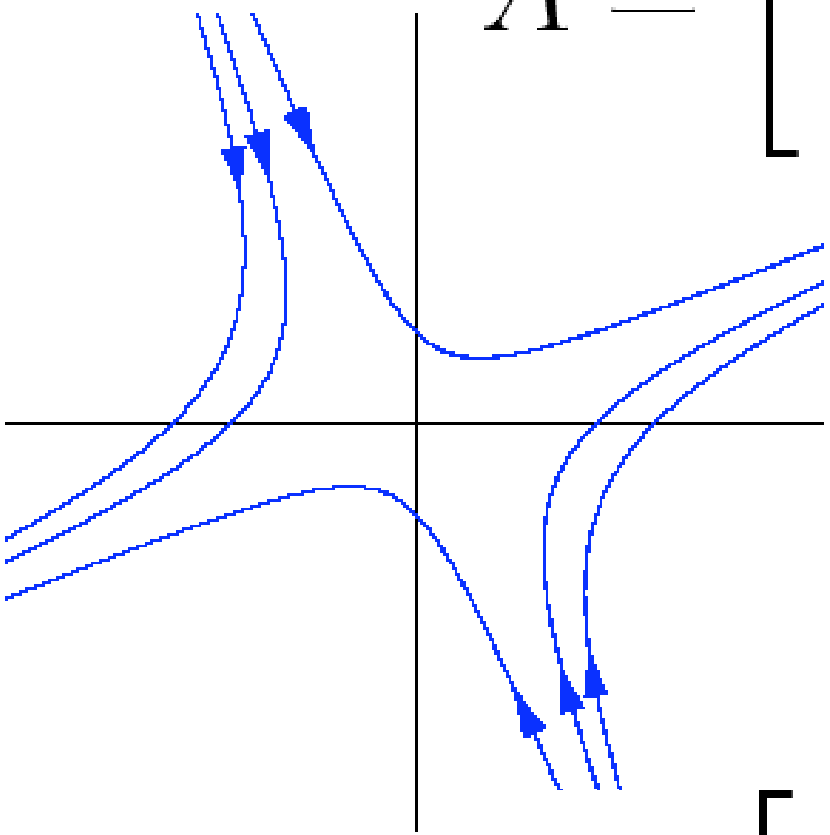
Solved on Blackboard

Asymptotically stable?

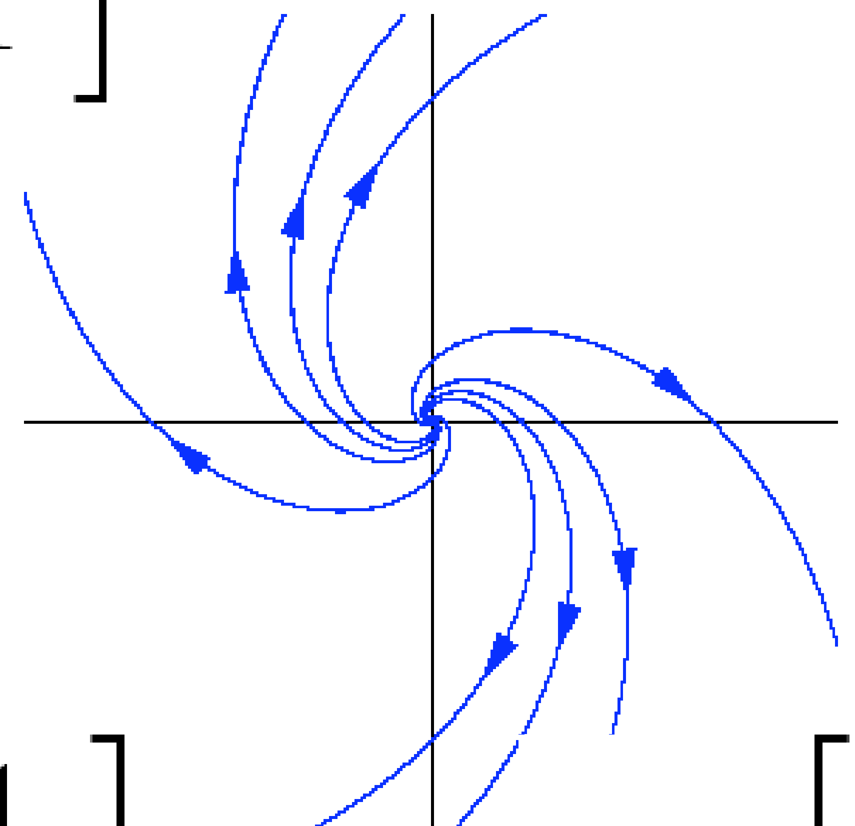
$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

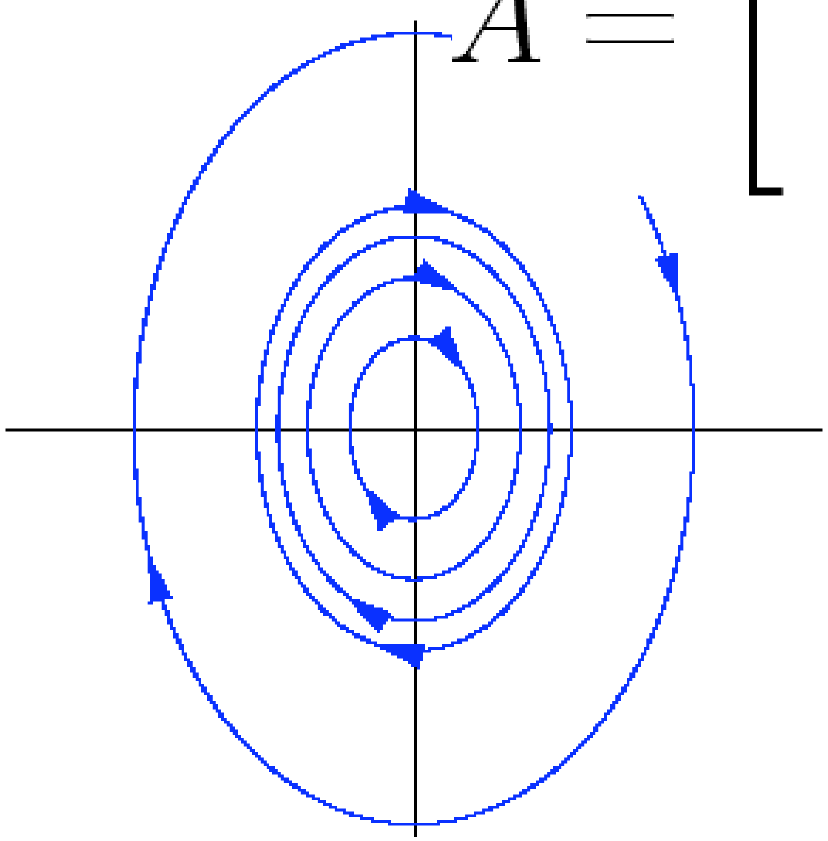
$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



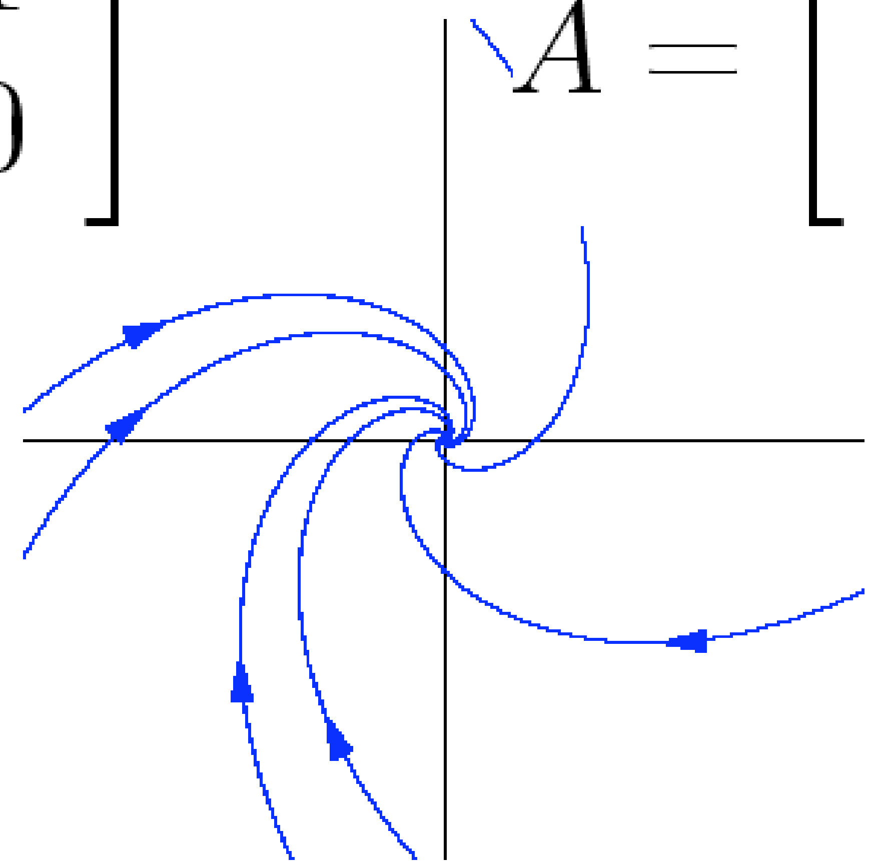
$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$



$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$



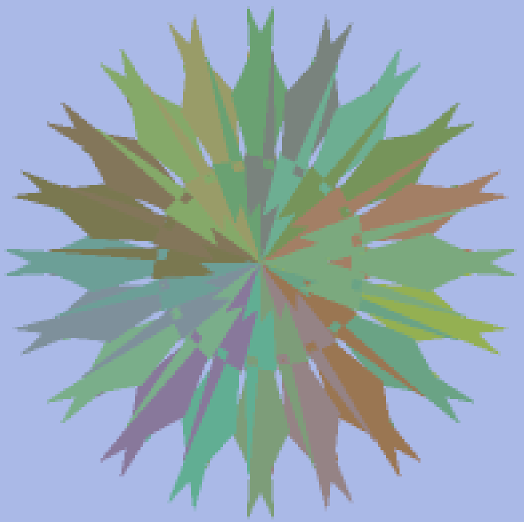
$$A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$



Caution!

Do not mix up results from discrete dynamical systems with the ones from differential equations!

III) Nonlinear differential equation

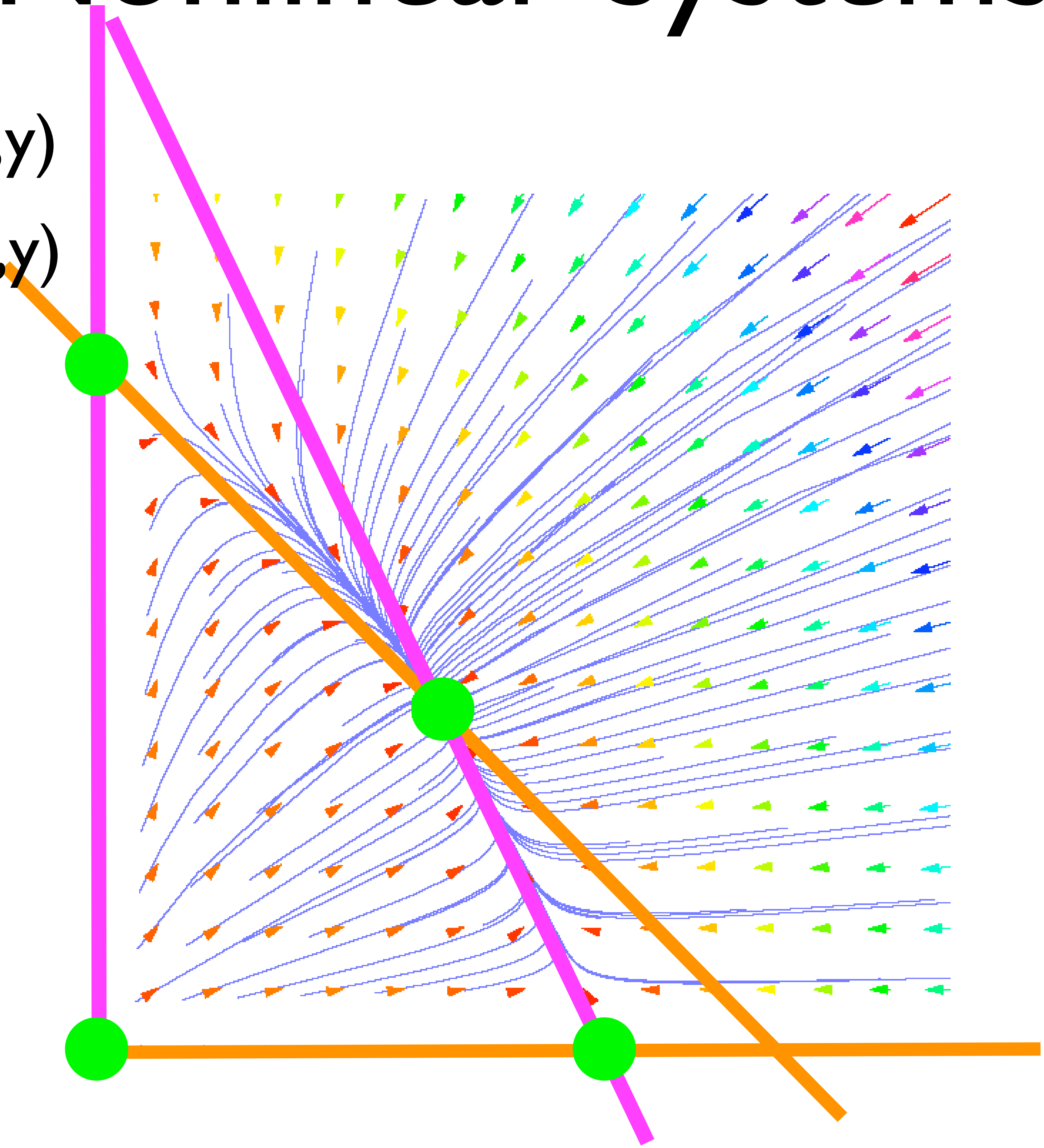


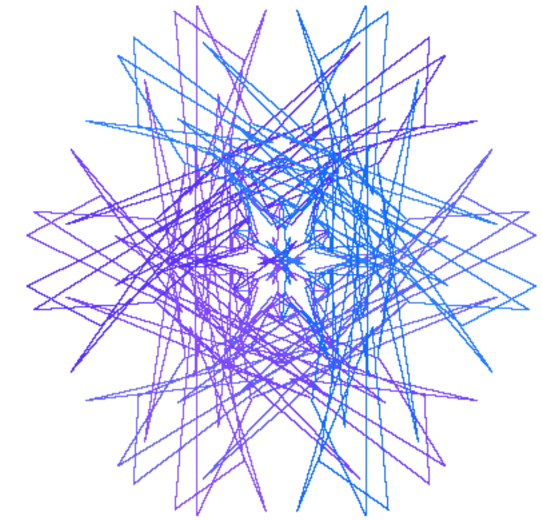
- Equilibrium points
- Nullclines
- Nature of equilibrium points
- Understanding phase space

Nonlinear Systems

$\dot{x}=f(x,y)$

$\dot{y}=g(x,y)$

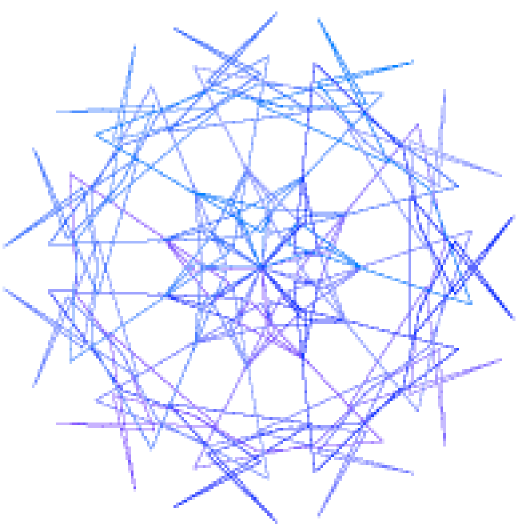


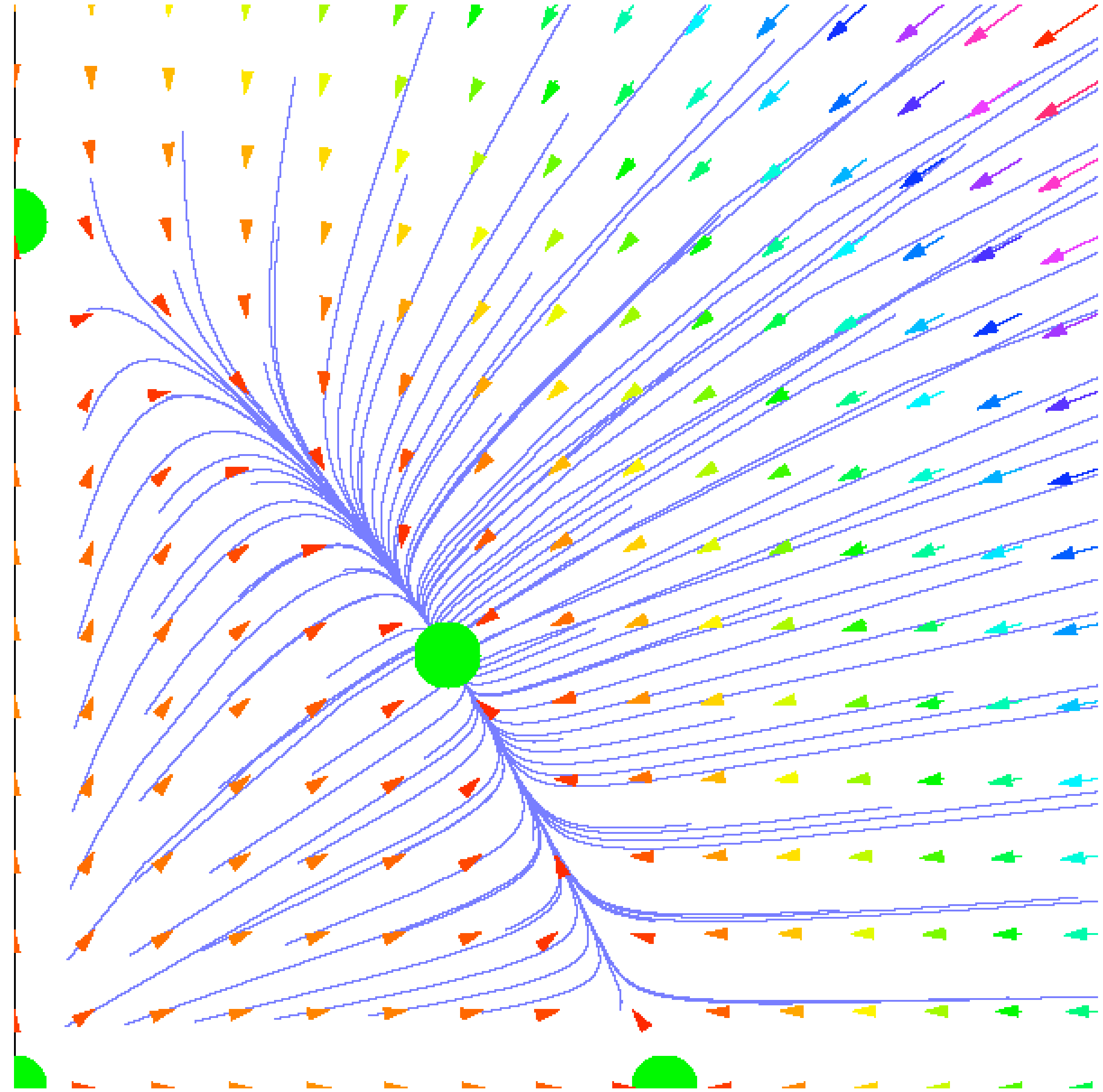


Null-clines

Equi-librium-points

Jacobean matrix

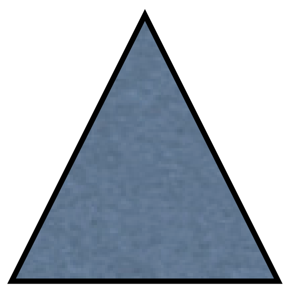




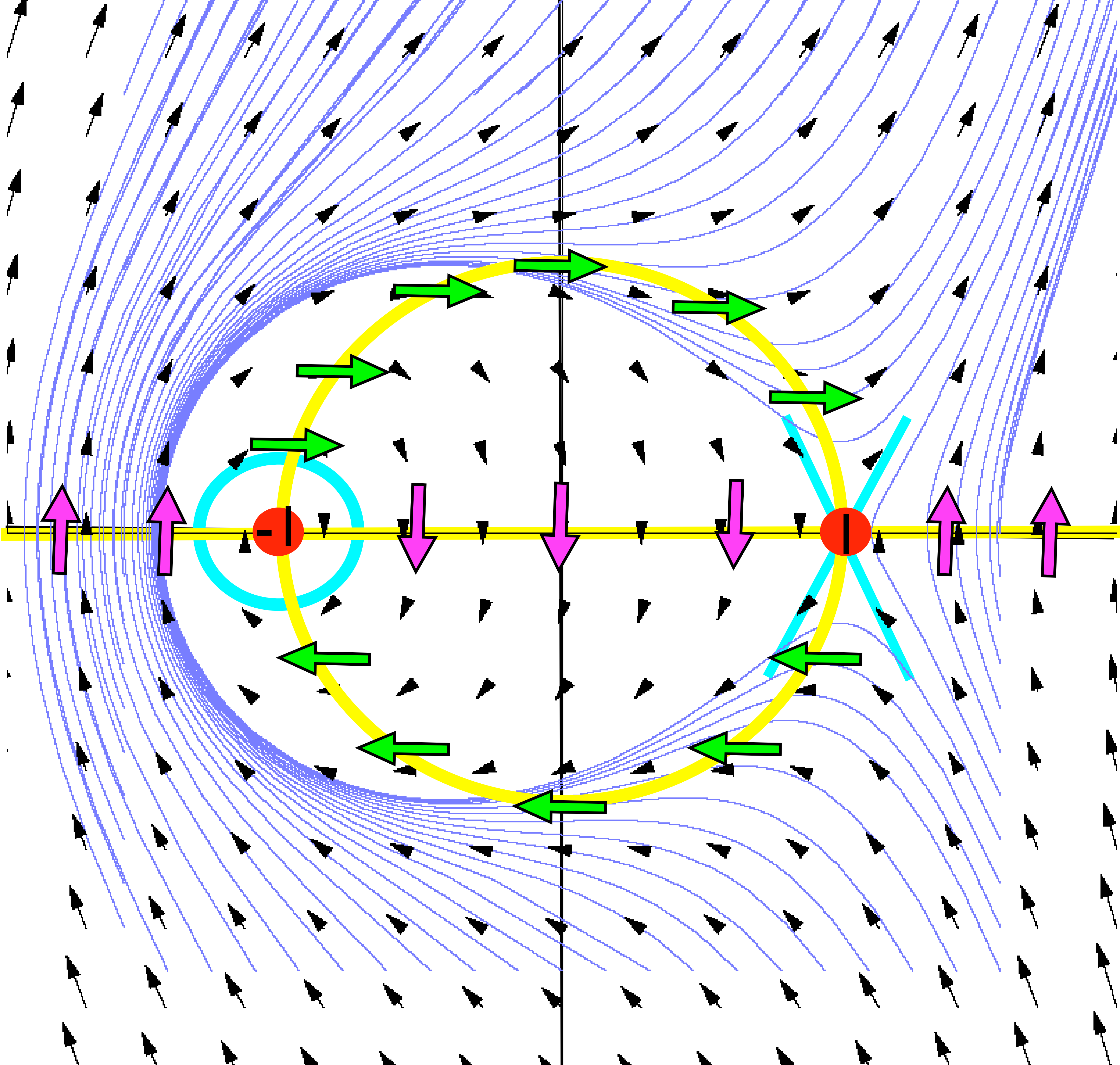
$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Example



Example Blackboard



IV) Higher Order Differential Equations

- Solving initial value problem p(D)f=g
- Homogenous problem p(D)f = 0
- Finding special solution to inhomogenous problem.

Nonhomogeneous differential equations

$$p(D)\, f(t) = g(t)$$

Three cases for second order differential equations:

1) Different real roots of p
2) Two identical real roots of p
3) Two complex roots of p

The method overview:

Problem:

$$f'' + 5 f = 2 \sin(3 x)$$
$$f(0)=1, f'(0)=2$$

Homogenous

$$f'' + 5 f = 0$$

$$\lambda^2 + 5 \lambda = 0$$
$$\lambda = i\sqrt{5} \quad -i\sqrt{5}$$

$$f(x) = a \sin(\sqrt{5}\, x) + b \cos(\sqrt{5}\, x)$$

Inhomogeneous

$$f'' + 5 f = 2 \sin(3 x)$$

$$f(x) = c \sin(3 x)$$
$$f'' + 5 f = c + 5c = 2$$
so $c = -1/2$

$$f(x) = -\sin(3x)/2$$

Initial conditions $f(0)=1$ implies $b=1$

$f'(0)=1$ implies $a=-3/(2\sqrt{5})$

Solution: $$f(x) = -3/(2\sqrt{5}) \sin(\sqrt{5}x) + \cos(\sqrt{5}x) - \sin(3x)/2$$

Four examples

Examples Blackboard

$$f' - 3f = \exp(t)$$

$$f'' - 6f' + 9f = \exp(t)$$

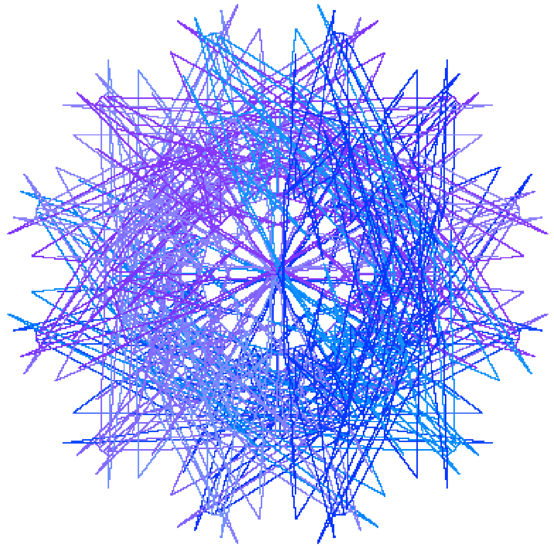
$$f'' + 9f = \exp(t)$$

$$f'' + 6f' + 8f = t$$

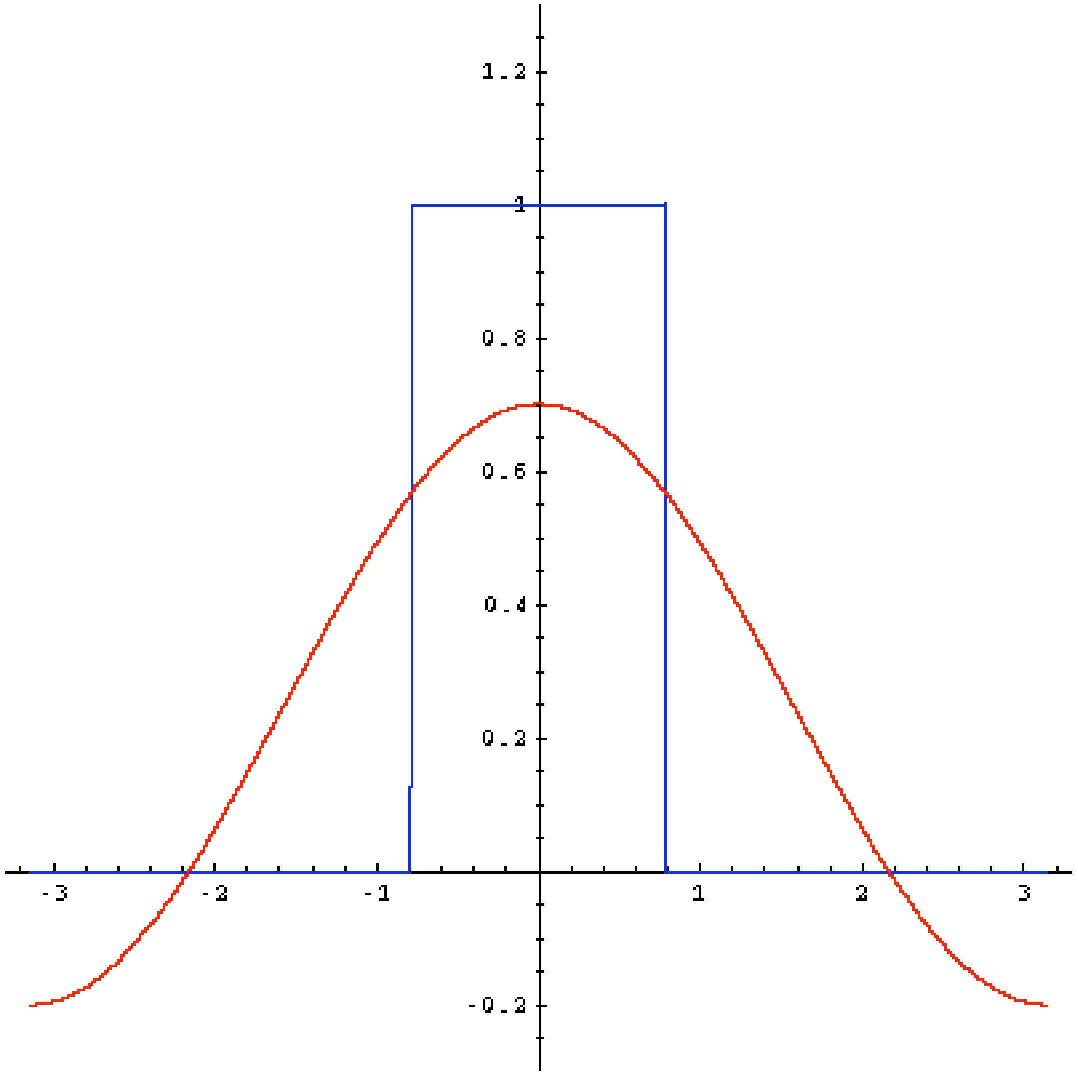
V) Fourier analysis

- Fourier Series
- Symmetry
- Parceval

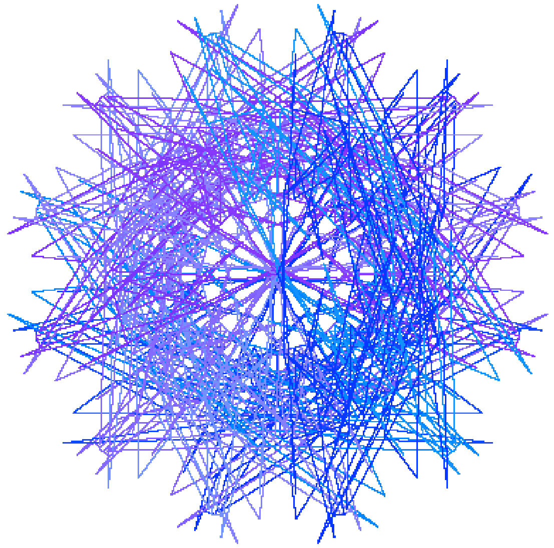
Fourier series



$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n \geq 1} a_n \cos(nx) + b_n \sin(nx)$$



Fourier coefficients

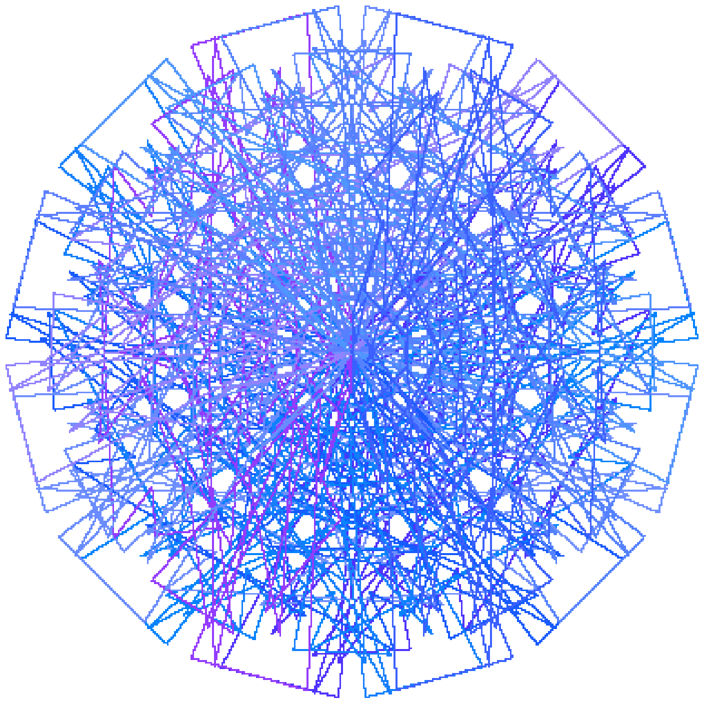


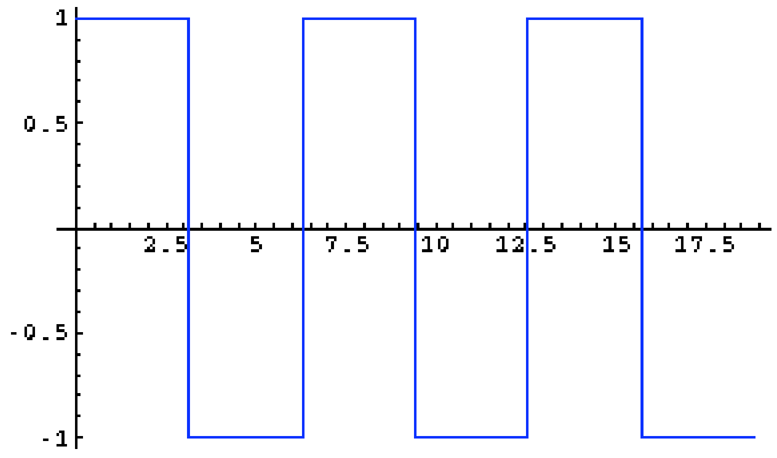
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\frac{1}{\sqrt{2}}\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

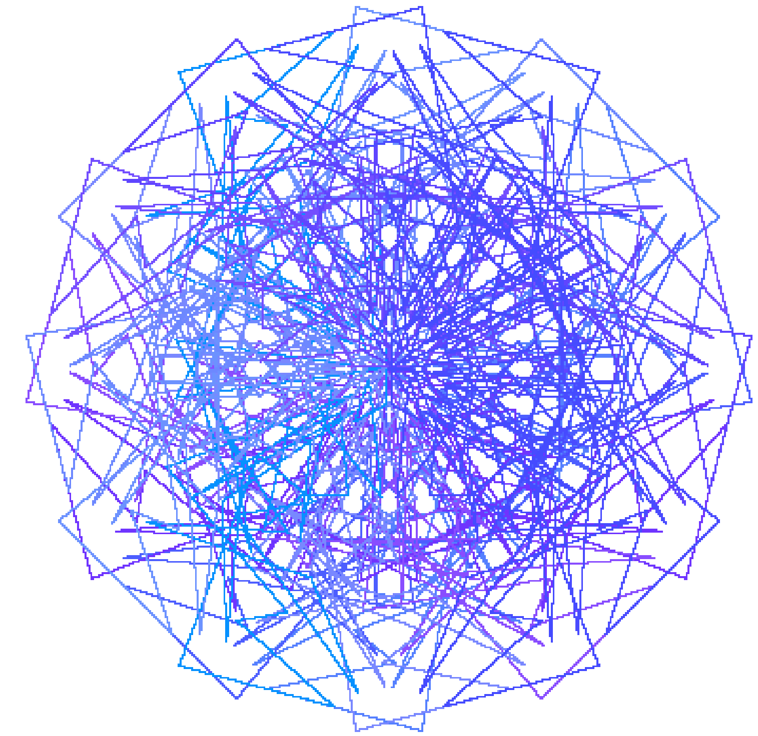
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

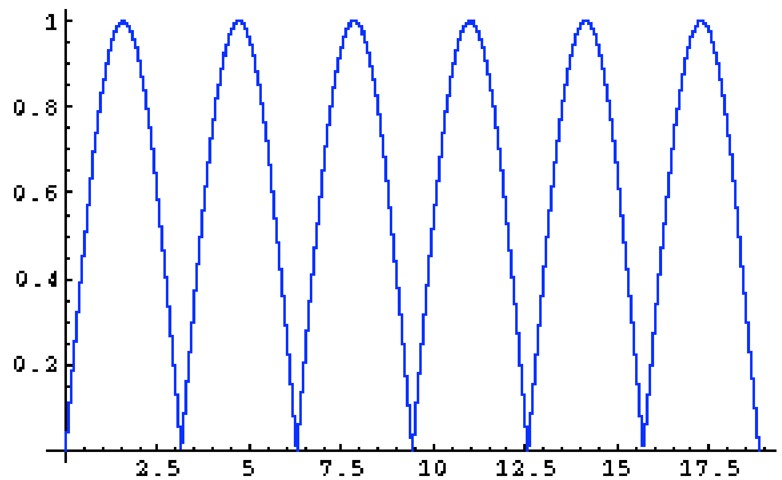
Sound synthesis I



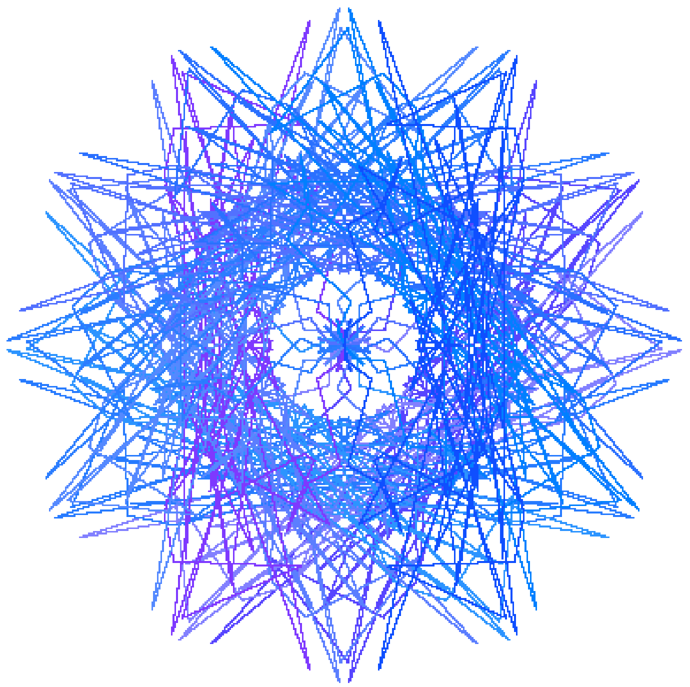


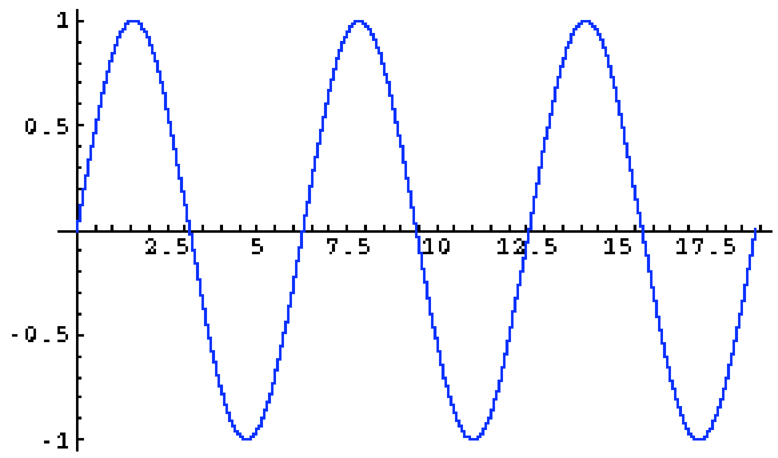
Sound synthesis II



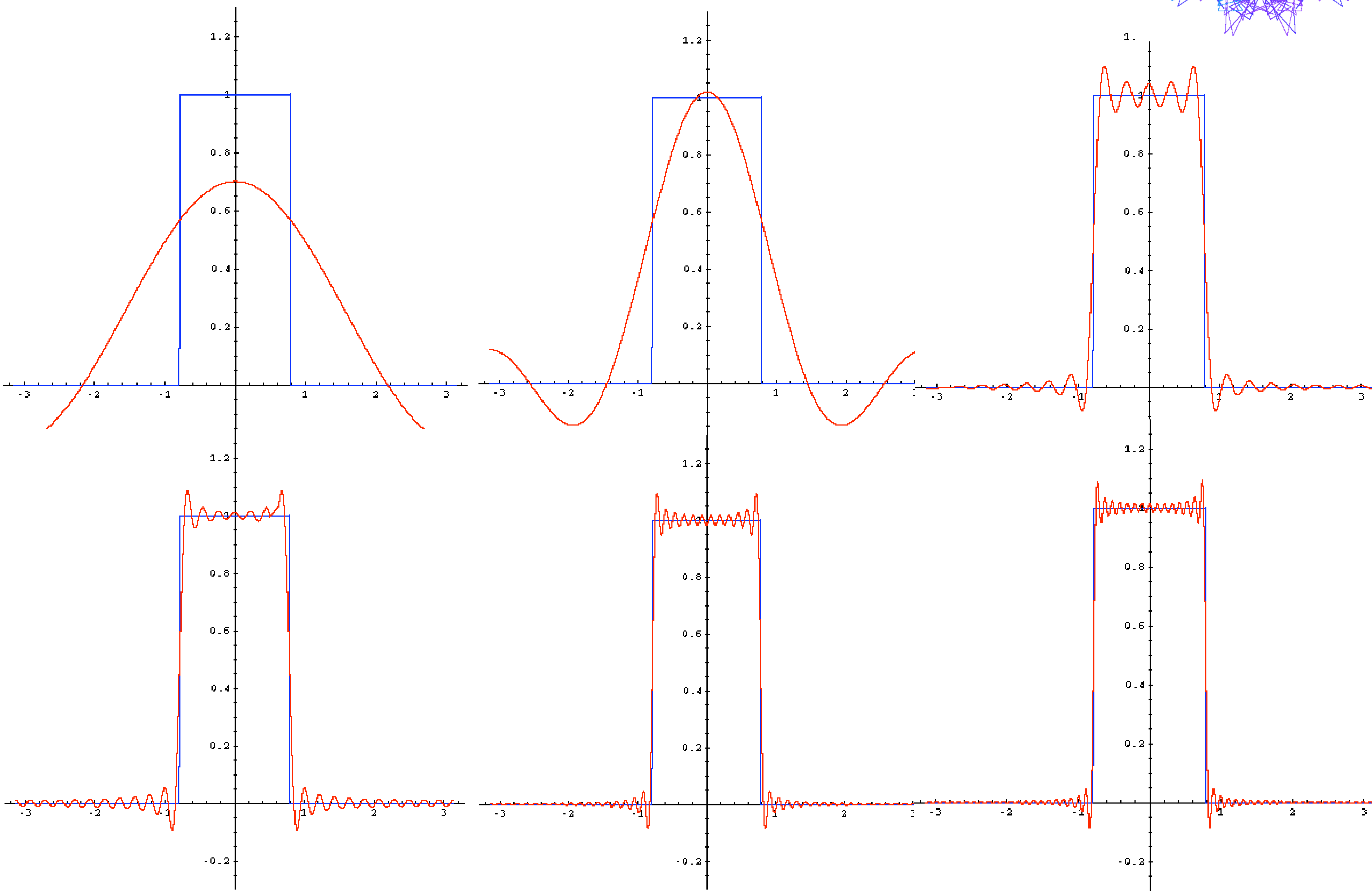


Sound synthesis III

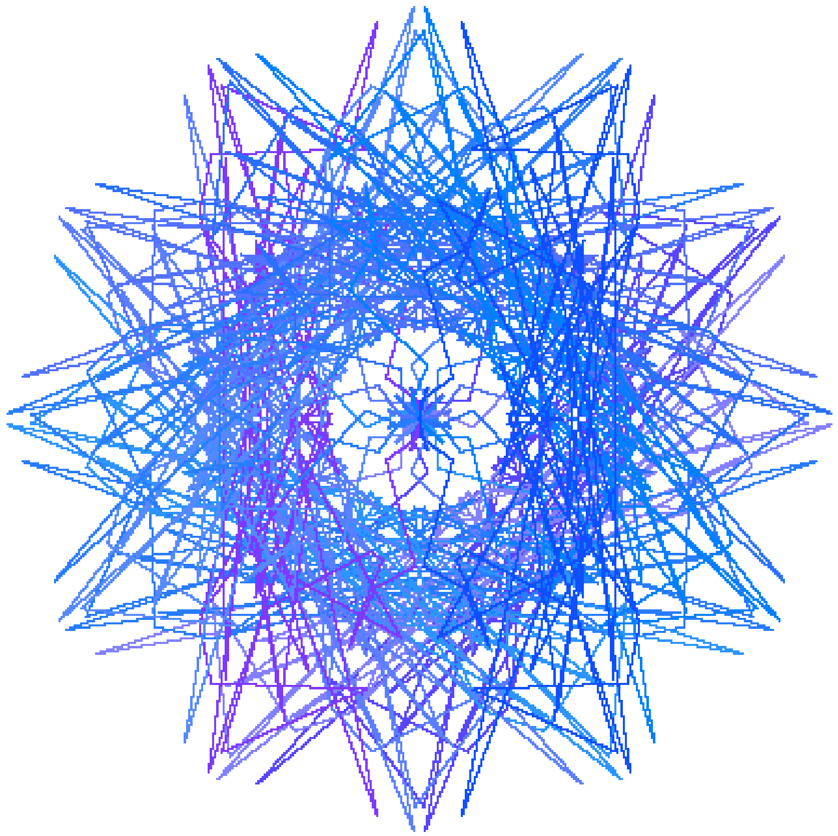


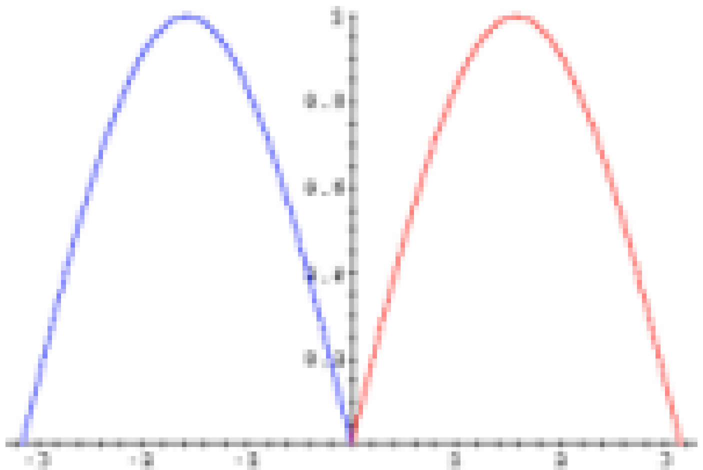


Fourier approximation

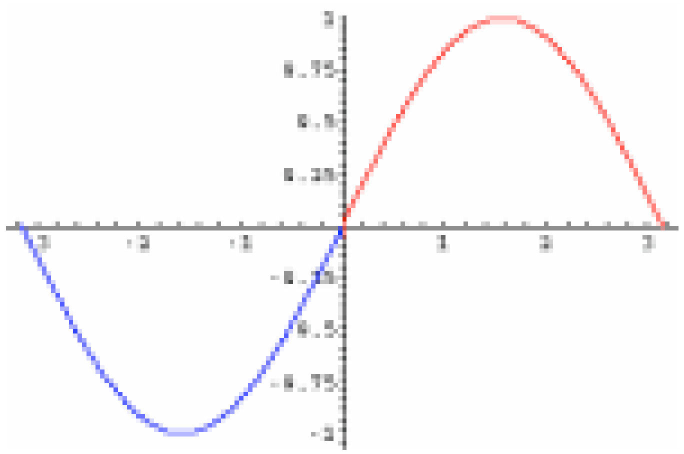


Even and Odd



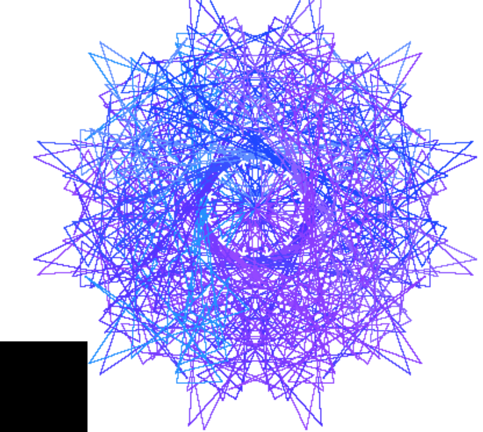


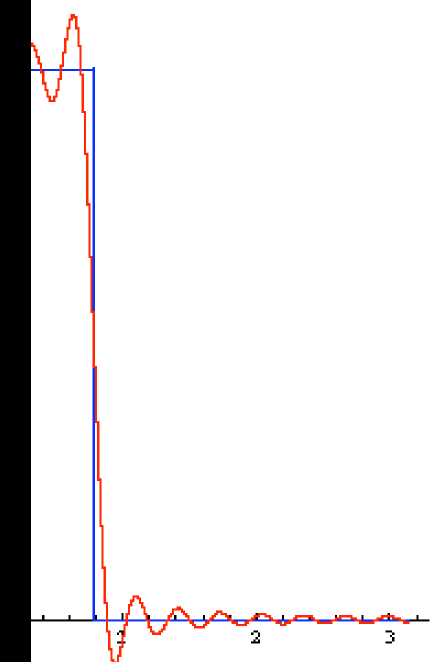
cos-series

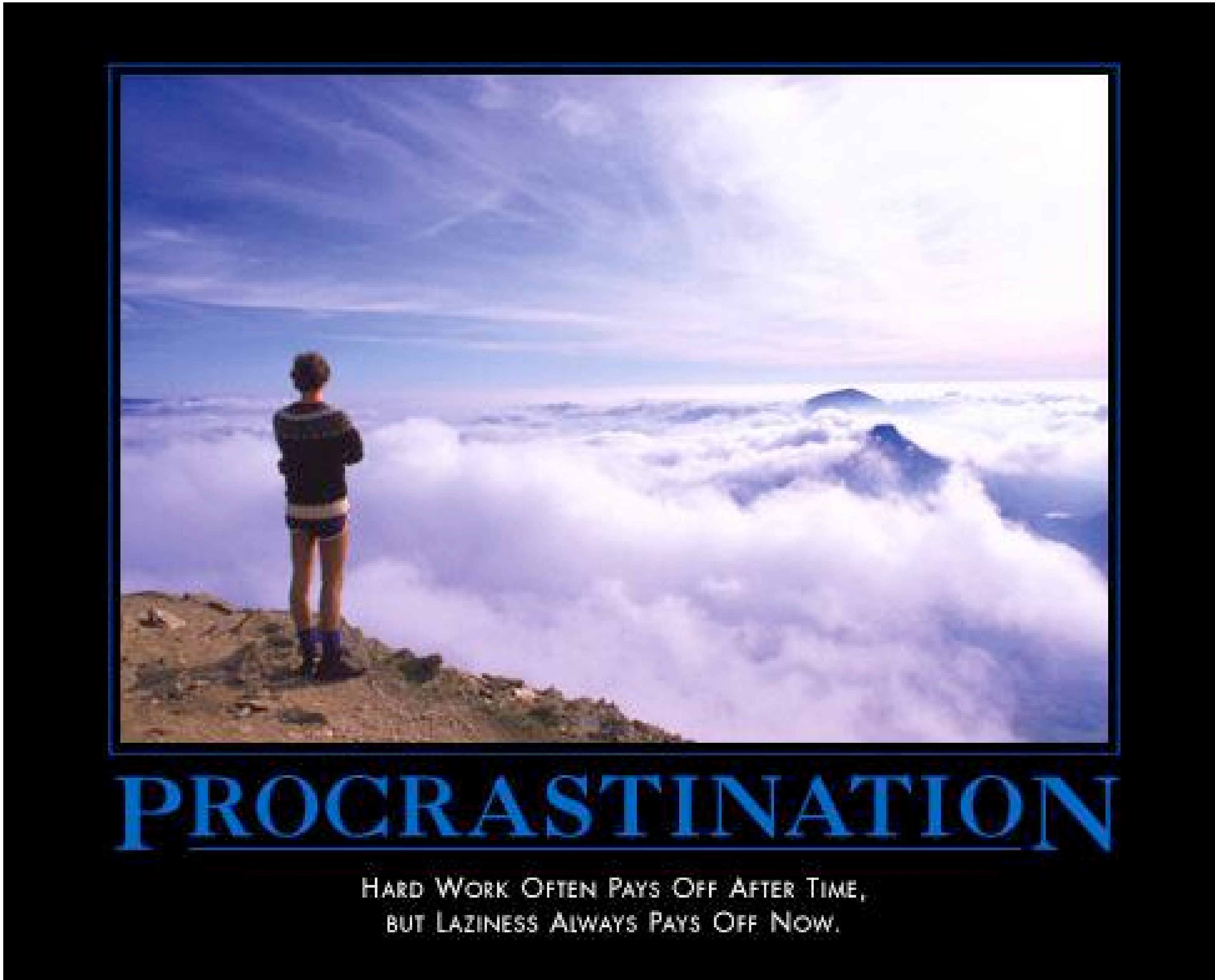


sin-series

Remember:





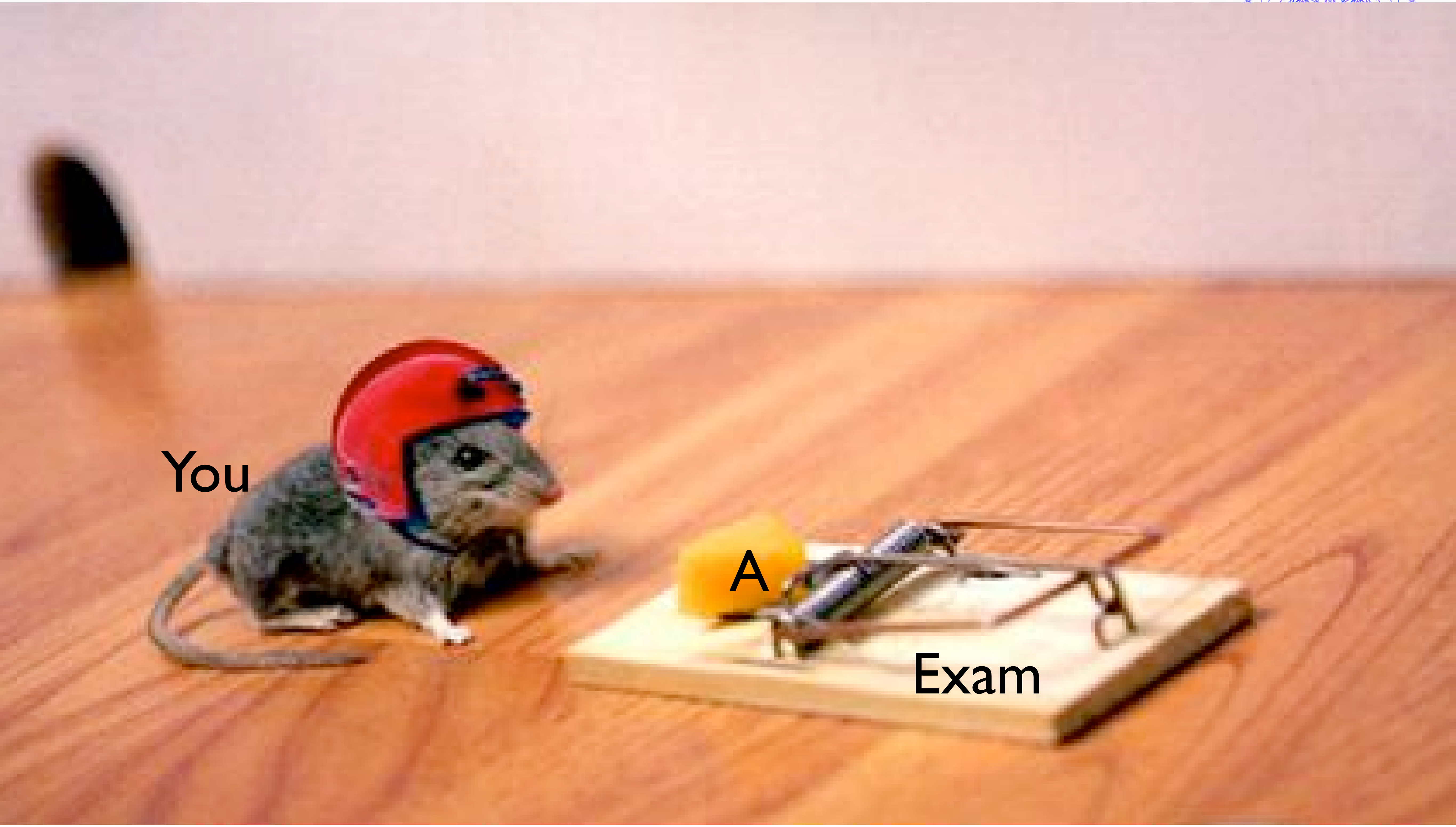

PROCRASTINATION
Hard Work Often Pays Off After Time,
but Laziness Always Pays Off Now.

Parseval Identity

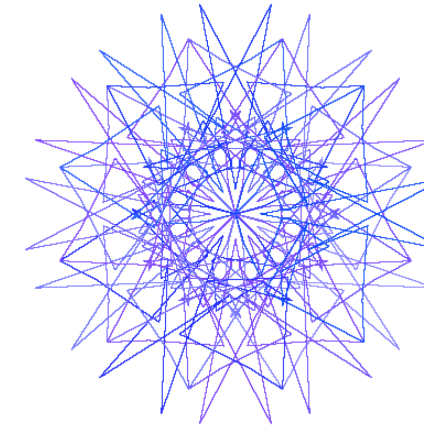
$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = ||f||^2$$

Some safety tips:



When doing Fourier integrals:



Seperate into even and odd parts!

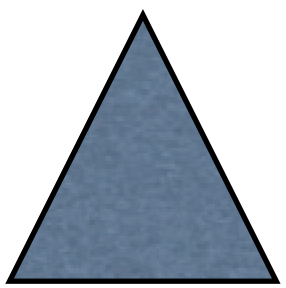
terms like sin(89 x) are already part of the Fourier decomposition.

Dont spend too much time with simplifying the result. Any correct result without integrals is good enough.

A music piece is just a function f(t).

We can manipulate this function: i.e. $T(f)(t)=f(-t)$

Example

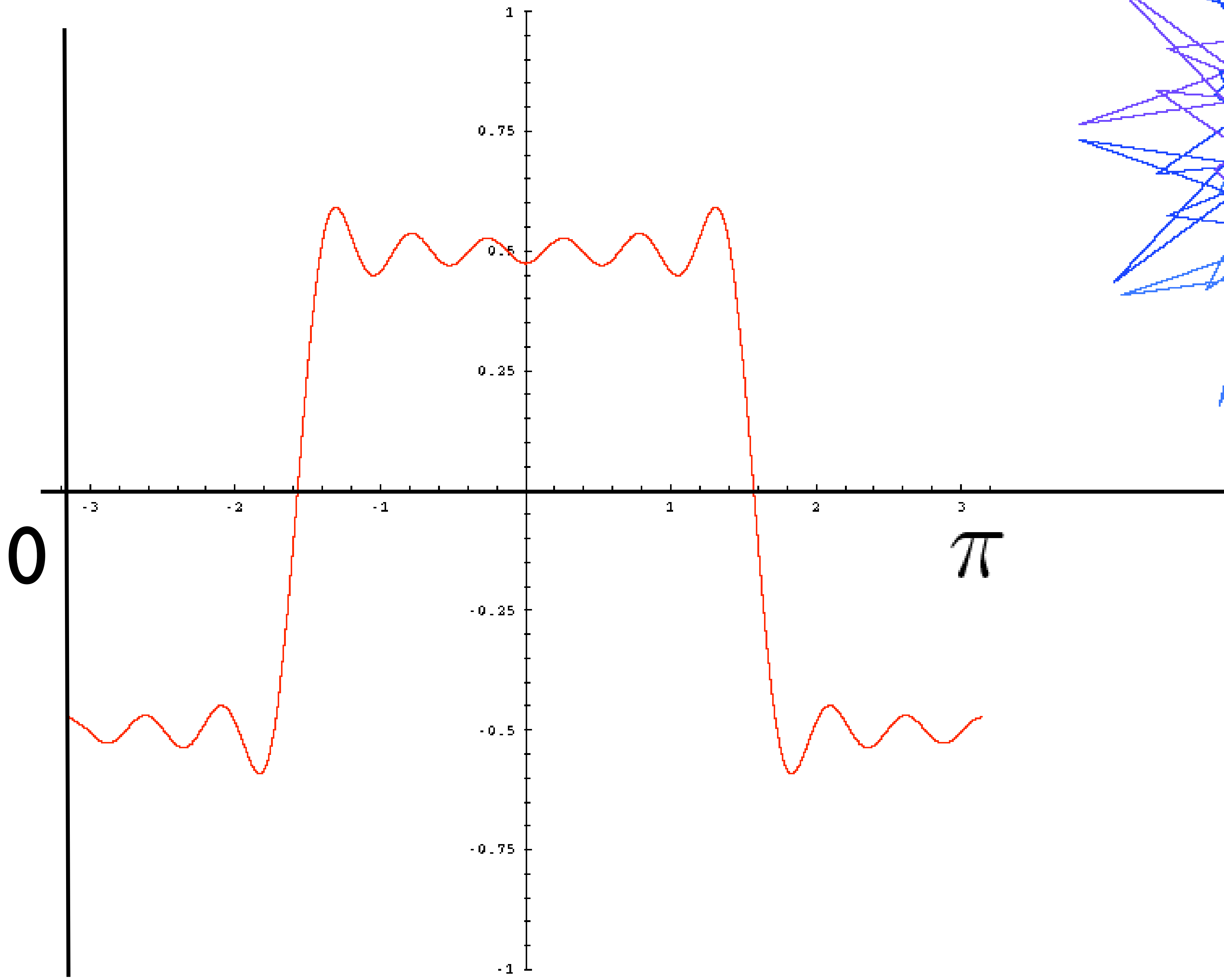


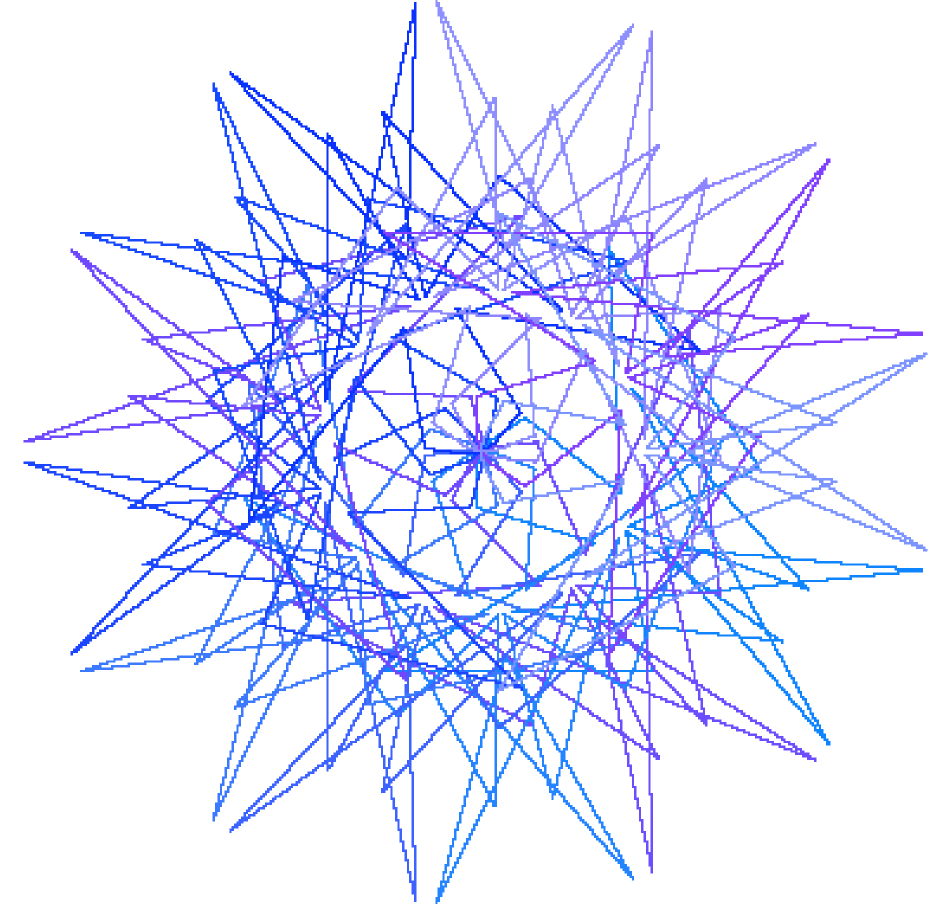
Example Blackboard

VI) Partial Differential equations

- Heat equation
- Wave equation

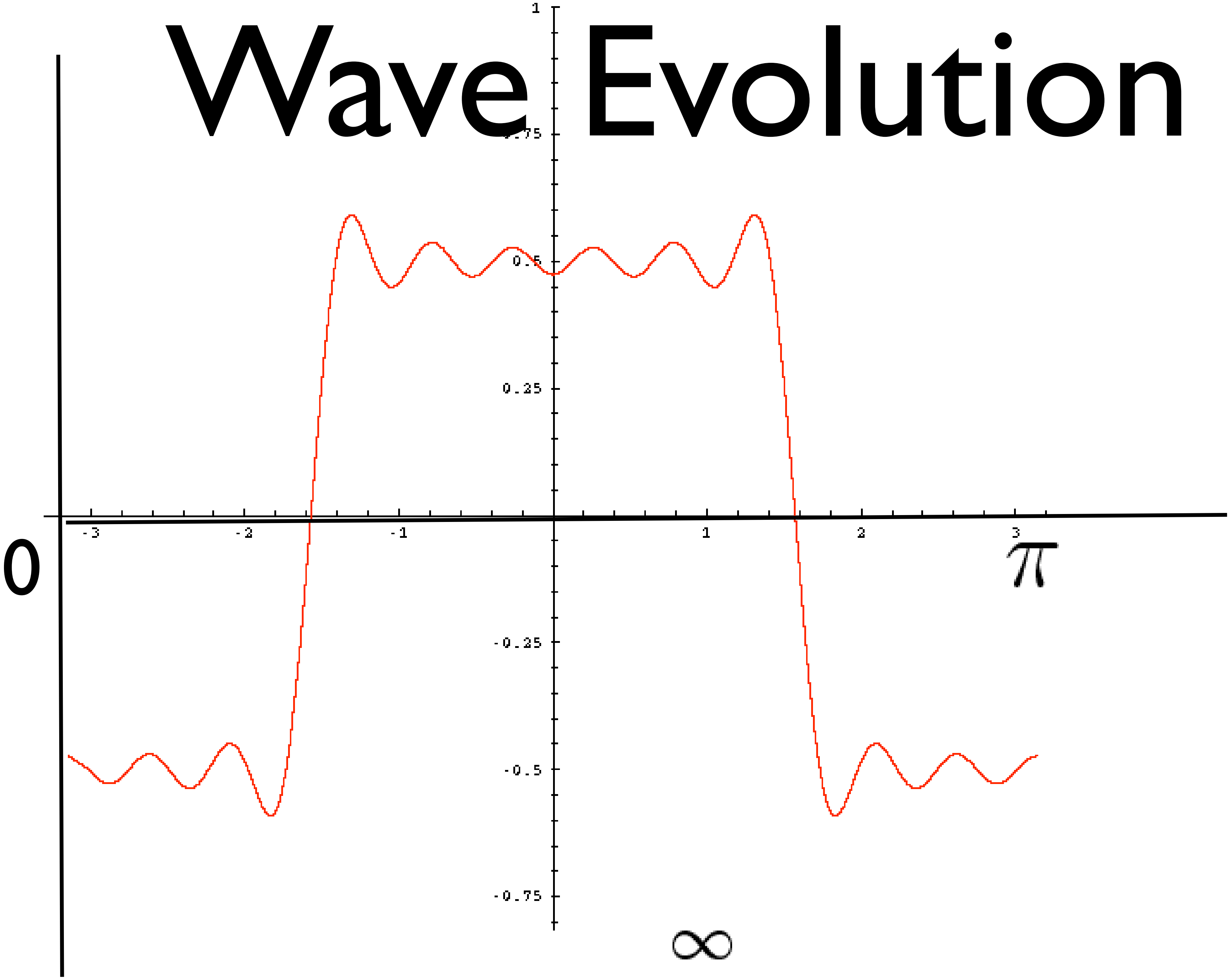
Heat Evolution

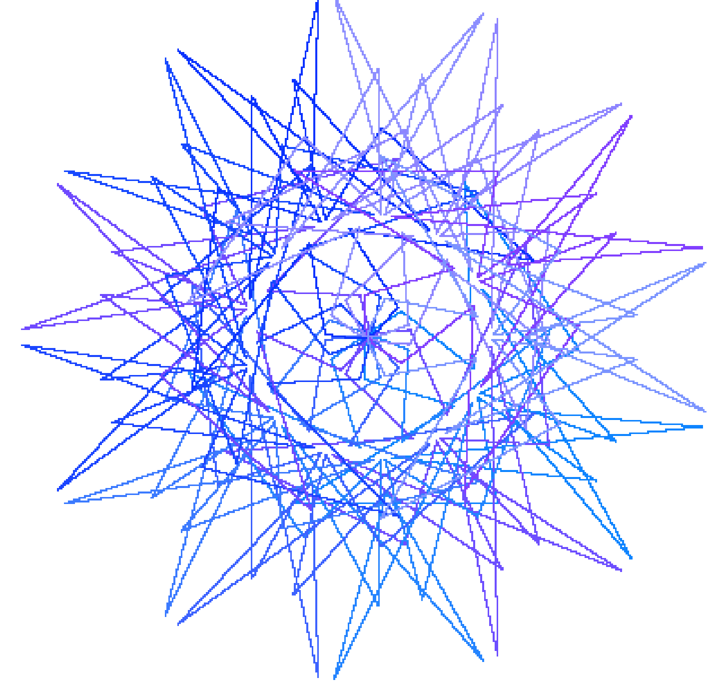




$$f(x,t) = \sum_{n=1}^{\infty} b_n e^{-\mu n^2 t} \sin(nx)$$

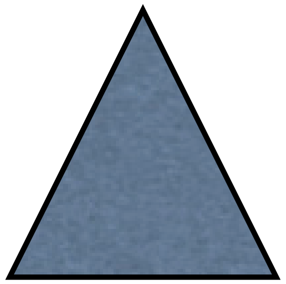
Wave Evolution





$$f(x,t) = \sum_{n=1}^{\infty} b_n \cos(cnt)\ \sin(nx) + \sum_{n=1}^{\infty} \tilde{b}_n \frac{\sin(cnt)}{nc}\ \sin(nx)$$

Example



Example Blackboard

Pro Memoria

- Exam is on January 20
- Review also midterm exams and homework
- Do the practice exams

and:

SCIENCE & SPACE

Huygens to plumb secrets of Saturn moon

By Michael Coren
CNN
Thursday, January 13, 2005 Posted: 2:12 PM EST (1912 GMT)

(CNN) -- The Huygens probe will plunge through the orange clouds of Saturn's moon Titan Friday, offering scientists their first glimpse of the mysterious moon.

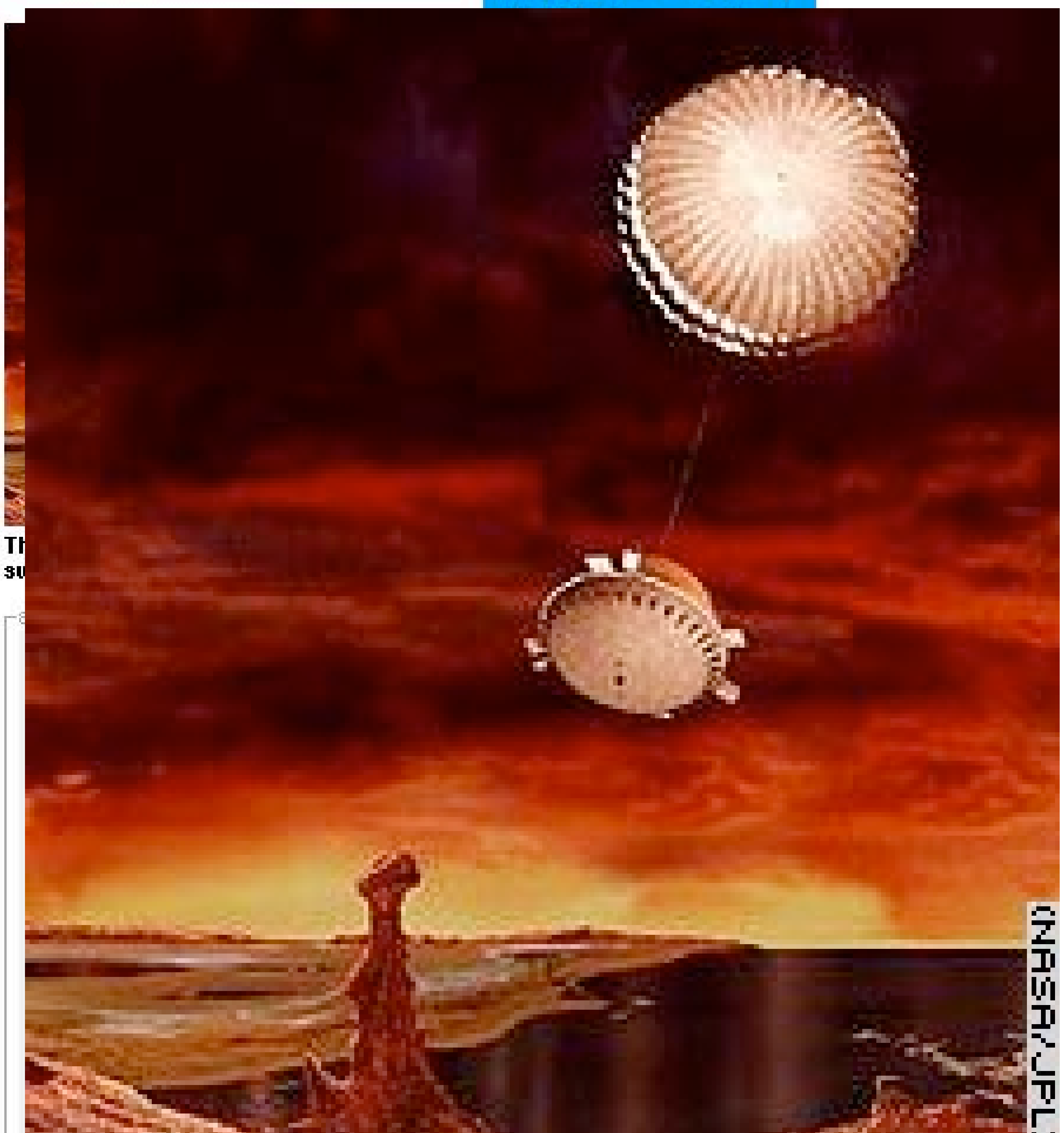
"It's going to be the most exotic place we've ever seen," said Candice Hansen, a scientist for the Cassini-Huygens mission. "We've never landed on the surface of an icy satellite. We know from our pictures that there are very different kinds of geological processes."

If all goes well, the saucer-shaped Huygens will enter the thick atmosphere of Titan Friday at about 5:13 a.m. (ET). The data should start trickling in about five hours later.

The Cassini-Huygens mission is an unprecedented $3.3-billion effort between NASA, the European Space Agency and Italy's space program to study Saturn and its 33 known moons. The two vehicles were launched together from Florida in 1997.

"The mission is to explore the entire Saturnian system in considerably greater detail than we have ever been able to do before: the atmosphere, the internal structure, the satellites, the rings, the magnetosphere," said Cassini program manager Bob Mitchell at NASA.

The Huygens probe, about the size of a Volkswagen-Beetle, has been spinning silently toward Titan since it detached from the Cassini spacecraft on December



Th
su