

Linear Algebra and Differential Equations

Math 21b

Harvard University

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These are some class notes distributed in the linear algebra course "Linear Algebra and Differential equations" taught in the Fall 2003. The course used the text book "Linear Algebra with Applications" by Otto Bretscher.

Some of the pages were intended as complements to the text and lectures in the years 2001-2003.

9/22/2003, LINEAR EQUATIONS

Math 21b, O. Knill

SYSTEM OF LINEAR EQUATIONS. A collection of linear equations is called a **system of linear equations**. An example is

$$\begin{cases} 3x - y - z = 0 \\ -x + 2y - z = 0 \\ -x - y + 3z = 9 \end{cases}.$$

It consists of three equations for three unknowns x, y, z . **Linear** means that no nonlinear terms like $x^2, x^3, xy, yz^3, \sin(x)$ etc. appear. A formal definition of linearity will be given later.

LINEAR EQUATION. More precisely, $ax + by = c$ is the general linear equation in two variables. $ax + by + cz = d$ is the general linear equation in three variables. The general **linear equation** in n variables is $a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$. Finitely many such equations form a **system of linear equations**.

SOLVING BY ELIMINATION.

Eliminate variables. In the example the first equation gives $z = 3x - y$. Putting this into the second and third equation gives

$$\begin{cases} -x + 2y - (3x - y) = 0 \\ -x - y + 3(3x - y) = 9 \end{cases}$$

or

$$\begin{cases} -4x + 3y = 0 \\ 8x - 4y = 9 \end{cases}.$$

The first equation gives $y = 4/3x$ and plugging this into the other equation gives $8x - 16/3x = 9$ or $8x = 27$ which means $x = 27/8$. The other values $y = 9/2, z = 45/8$ can now be obtained.

SOLVE BY SUITABLE SUBTRACTION.

Addition of equations. If we subtract the third equation from the second, we get $3y - 4z = -9$ and add three times the second equation to the first, we get $5y - 4z = 0$. Subtracting this equation to the previous one gives $-2y = -9$ or $y = 2/9$.

SOLVE BY COMPUTER.

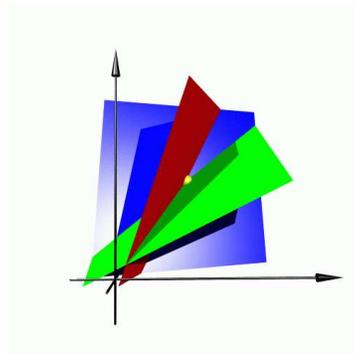
Use the computer. In Mathematica:

$$\text{Solve}\{3x - y - z == 0, -x + 2y - z == 0, -x - y + 3z == 9\}, \{x, y, z\}.$$

But what did Mathematica do to solve this equation? We will look in this course at some efficient algorithms.

GEOMETRIC SOLUTION.

Each of the three equations represents a plane in three-dimensional space. Points on the first plane satisfy the first equation. The second plane is the solution set to the second equation. To satisfy the first two equations means to be on the intersection of these two planes which is here a line. To satisfy all three equations, we have to intersect the line with the plane representing the third equation which is a point.



LINE, PLANES, HYPERPLANES.

The set of points in the plane satisfying $ax + by = c$ form a **line**.

The set of points in space satisfying $ax + by + cz = d$ form a **plane**.

The set of points in n -dimensional space satisfying $a_1x_1 + \dots + a_nx_n = a_0$ define a set called a **hyperplane**.

RIDDLES:

"15 kids have bicycles or tricycles. Together they count 37 wheels. How many have bicycles?"

Solution. With x bicycles and y tricycles, then $x + y = 15, 2x + 3y = 37$. The solution is $x = 8, y = 7$.

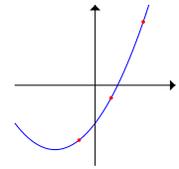
"Tom, the brother of Carry has twice as many sisters as brothers while Carry has equal number of sisters and brothers. How many kids is there in total in this family?"

Solution If there are x brothers and y sisters, then Tom has y sisters and $x - 1$ brothers while Carry has x brothers and $y - 1$ sisters. We know $y = 2(x - 1), x = y - 1$ so that $x + 1 = 2(x - 1)$ and so $x = 3, y = 4$.

INTERPOLATION.

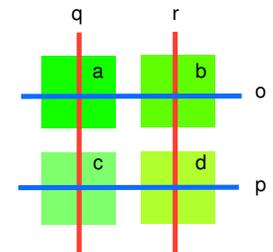
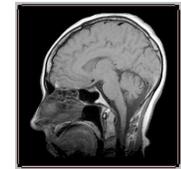
Find the equation of the parabola which passes through the points $P = (0, -1), Q = (1, 4)$ and $R = (2, 13)$.

Solution. Assume the parabola is $ax^2 + bx + c = 0$. So, $c = -1, a + b + c = 4, 4a + 2b + c = 13$. Elimination of c gives $a + b = 5, 4a + 2b = 14$ so that $2b = 6$ and $b = 3, a = 2$. The parabola has the equation $2x^2 + 3x - 1 = 0$



TOMOGRAPHY

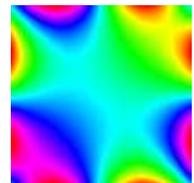
Here is a toy example of a problem one has to solve for magnetic resonance imaging (MRI), which makes use of the absorption and emission of energy in the radio frequency range of the electromagnetic spectrum.



Assume we have 4 hydrogen atoms whose nuclei are excited with intensity a, b, c, d . We measure the spin echo intensity in 4 different directions. $3 = a + b, 7 = c + d, 5 = a + c$ and $5 = b + d$. What is a, b, c, d ? Solution: $a = 2, b = 1, c = 3, d = 4$. However, also $a = 0, b = 3, c = 5, d = 2$ solves the problem. This system has not a unique solution even so there are 4 equations and 4 unknowns. A good introduction into MRI can be found online at (<http://www.cis.rit.edu/htbooks/mri/inside.htm>).

INCONSISTENT. $x - y = 4, y + z = 5, x + z = 6$ is a system with no solutions. It is called **inconsistent**.

EQUILIBRIUM. As an example of a system with many variables, consider a drum modeled by a fine net. The heights at each interior node needs the average the heights of the 4 neighboring nodes. The height at the boundary is fixed. With n^2 nodes in the interior, we have to solve a system of n^2 equations. For example, for $n = 2$ (see left), the $n^2 = 4$ equations are $x_{11} = a_{21} + a_{12} + x_{21} + x_{12}, x_{12} = x_{11} + x_{13} + x_{22} + x_{22}, x_{21} = x_{31} + x_{11} + x_{22} + a_{43}, x_{22} = x_{12} + x_{21} + a_{43} + a_{34}$. To the right, we see the solution to a problem with $n = 300$, where the computer had to solve a system with 90'000 variables.



LINEAR OR NONLINEAR?

a) **The ideal gas law** $PV = nKT$ for the P, V, T , the pressure, volume and temperature of the gas.

b) **The Hook law** $F = k(x - x_0)$ relates the force F pulling a string at position x which is relaxed at x_0 .

MATRIX FORMULATION. Consider the system of linear equations. The system can be written as $A\vec{x} = \vec{b}$, where A is a **matrix** (called **coefficient matrix**) and \vec{x} and \vec{b} are **vectors**.

$$\begin{cases} 3x - y - z = 0 \\ -x + 2y - z = 0 \\ -x - y + 3z = 9 \end{cases}$$

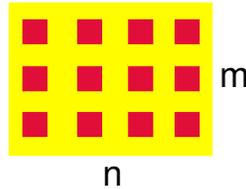
$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}.$$

$((A\vec{x})_i)$ is the dot product of the i 'th row with \vec{x} .

We also look at the **augmented matrix** where one puts separators for clarity reasons.

$$B = \left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & 9 \end{array} \right].$$

MATRIX "JARGON". A rectangular array of numbers is called a **matrix**. If the matrix has m **rows** and n **columns**, it is called a $m \times n$ matrix. A matrix with one column only is called a **column vector**, a matrix with one row a **row vector**. The entries of a matrix are denoted by a_{ij} , where i is the row and j is the column. In the case of the linear equation above, the matrix A is a square matrix and the augmented matrix B above is a 3×4 matrix.



GAUSS-JORDAN ELIMINATION. Gauss-Jordan Elimination is a process, where successive subtraction of multiples of other rows or scaling brings the matrix into **reduced row echelon form**. The elimination process consists of three possible steps which are called **elementary row operations**:

- **Swap** two rows.
- **Divide** a row by a scalar
- **Subtract** a multiple of a row from another row.

The process transfers a given matrix A into a new matrix $\text{rref}(A)$

REDUCED ECHELON FORM. A matrix is called in **reduced row echelon form**

- 1) if a row has nonzero entries, then the first nonzero entry is 1. (**leading one**)
- 2) if a column contains a leading 1, then the other column entries are 0.
- 3) if a row has a leading 1, then every row above has leading 1's to the left.

Pro memoriam: **Leaders like to be number one, are lonely and want other leaders above to their left.**

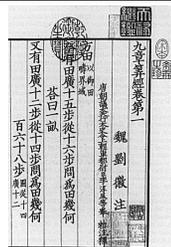
RANK. The number of leading 1 in $\text{rref}(A)$ is called the rank of A .

SOLUTIONS OF LINEAR EQUATIONS. If $Ax = b$ is a linear system of equations with m equations and n unknowns, then A is a $m \times n$ matrix. We have the following three possibilities:

- **Exactly one solution.** There is a leading 1 in each row but not in the last row.
- **Zero solutions.** There is a leading 1 in the last row.
- **Infinitely many solutions.** There are rows without leading 1 and no leading 1 is in the last row.

JIUZHANG SUANSHU. The technique of successively eliminating variables from systems of linear equations is called **Gauss elimination** or **Gauss Jordan elimination** and appeared already in the Chinese manuscript "Jiuzhang Suanshu" ('Nine Chapters on the Mathematical art'). The manuscript appeared around 200 BC in the Han dynasty and was probably used as a textbook. For more history of Chinese Mathematics, see

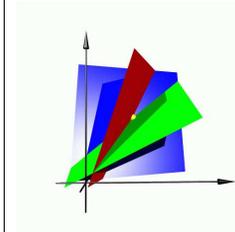
<http://aleph0.clarku.edu/~djoyce/mathhist/china.html>.



EXAMPLES. The reduced echelon form of the augmented matrix B determines on how many solutions the linear system $Ax = b$ has.



THE GOOD (1 solution)



$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 2 & 1 & -1 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 3 & -3 & -12 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & -1 & -4 \end{array} \right]$$

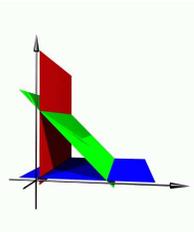
$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -3 & -6 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Rank(A) = 3, Rank(B) = 3.

THE BAD (0 solution)



$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 0 & 3 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & -2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & -7 \end{array} \right]$$

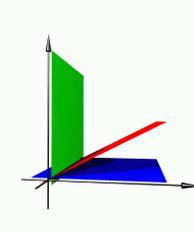
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -9 \end{array} \right]$$

Rank(A) = 2, Rank(B) = 3.

THE UGLY (∞ solutions)



$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 2 \\ 1 & -1 & 1 & 5 \\ 1 & 0 & 3 & 7 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & 7 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 3 & 7 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Rank(A) = 2, Rank(B) = 2.

JORDAN. The German geodesist Wilhelm **Jordan** (1842-1899) applied the Gauss-Jordan method to finding squared errors to work on surveying. (An other "Jordan", the French Mathematician Camille Jordan (1838-1922) worked on linear algebra topics also (Jordan form) and is often mistakenly credited with the Gauss-Jordan process.)

GAUSS. **Gauss** developed Gaussian elimination around 1800 and used it to solve least squares problems in celestial mechanics and later in geodesic computations. In 1809, Gauss published the book "Theory of Motion of the Heavenly Bodies" in which he used the method for solving astronomical problems. One of Gauss successes was the prediction of an asteroid orbit using linear algebra.

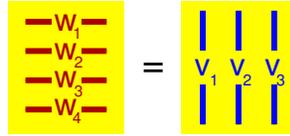


CERES. On 1. January of 1801, the Italian astronomer Giuseppe Piazzi (1746-1826) discovered **Ceres**, the first and until 2001 the largest known asteroid in the solar system. (A new found object called 2001 KX76 is estimated to have a 1200 km diameter, half the size of Pluto) Ceres is a rock of 914 km diameter. (The pictures Ceres in infrared light). Gauss was able to predict the orbit of Ceres from a few observations. By parameterizing the orbit with parameters and solving a linear system of equations (similar to one of the homework problems, where you will fit a cubic curve from 4 observations), he was able to derive the orbit parameters.



MATRIX. A rectangular array of numbers is called a **matrix**.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$



A matrix with m rows and n columns is called a $m \times n$ matrix. A matrix with one column is a **column vector**. The entries of a matrix are denoted a_{ij} , where i is the row number and j is the column number.

ROW AND COLUMN PICTURE. Two interpretations

$$A\vec{x} = \begin{bmatrix} -\vec{w}_1- \\ -\vec{w}_2- \\ \cdots \\ -\vec{w}_m- \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix} = \begin{bmatrix} \vec{w}_1 \cdot \vec{x} \\ \vec{w}_2 \cdot \vec{x} \\ \cdots \\ \vec{w}_m \cdot \vec{x} \end{bmatrix}$$



"Row and Column at Harvard"

$$A\vec{x} = \begin{bmatrix} | & | & \cdots & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m = \vec{b}$$

Row picture: each b_i is the dot product of a row vector \vec{w}_i with \vec{x} .
Column picture: \vec{b} is a sum of scaled column vectors \vec{v}_j .

EXAMPLE. The system of linear equations

$$\begin{cases} 3x - 4y - 5z = 0 \\ -x + 2y - z = 0 \\ -x - y + 3z = 9 \end{cases}$$

is equivalent to $A\vec{x} = \vec{b}$, where A is a **coefficient matrix** and \vec{x} and \vec{b} are **vectors**.

$$A = \begin{bmatrix} 3 & -4 & -5 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix}$$

The **augmented matrix** (separators for clarity)

$$B = \left[\begin{array}{ccc|c} 3 & -4 & -5 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & 9 \end{array} \right]$$

In this case, the row vectors of A are

$$\vec{w}_1 = \begin{bmatrix} 3 & -4 & -5 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

The column vectors are

$$\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix}$$

Row picture:

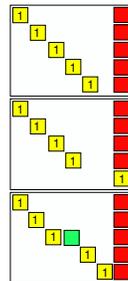
$$0 = b_1 = \begin{bmatrix} 3 & -4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Column picture:

$$\begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix}$$

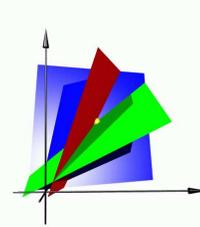
SOLUTIONS OF LINEAR EQUATIONS. A system $A\vec{x} = \vec{b}$ with m equations and n unknowns is defined by the $m \times n$ matrix A and the vector \vec{b} . The row reduced matrix $\text{rref}(B)$ of B determines the number of solutions of the system $Ax = b$. There are three possibilities:

- **Consistent: Exactly one solution.** There is a leading 1 in each row but none in the last row of B .
- **Inconsistent: No solutions.** There is a leading 1 in the last row of B .
- **Infinitely many solutions.** There are rows of B without leading 1. If $m < n$ (less equations than unknowns), then there are either zero or infinitely many solutions.

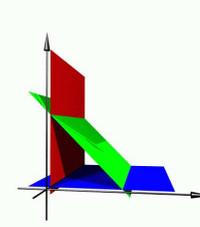


The **rank** $\text{rank}(A)$ of a matrix A is the number of leading ones in $\text{rref}(A)$.

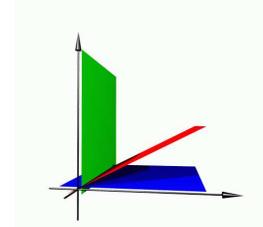
(exactly one solution)



(no solution)



(infinitely many solutions)

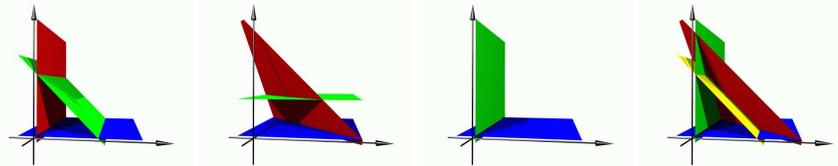


MURPHYS LAW.

"If anything can go wrong, it will go wrong".
 "If you are feeling good, don't worry, you will get over it!"
 "Whenever you do Gauss-Jordan elimination, you screw up during the first couple of steps."

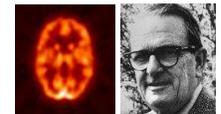


MURPHYS LAW IS TRUE. Two equations could contradict each other. Geometrically this means that the two planes do not intersect. This is possible if they are parallel. Even without two planes being parallel, it is possible that there is no intersection between all three of them. Also possible is that we don't have enough equations (for example because two equations are the same) and that there are many solutions. Furthermore, we can have too many equations and the four planes would not intersect.



RELEVANCE OF EXCEPTIONAL CASES. There are important applications, where "unusual" situations happen: For example in medical tomography, systems of equations appear which are "ill posed". In this case one has to be careful with the method.

The linear equations are then obtained from a method called the **Radon transform**. The task for finding a good method had led to a Nobel prize in Medicis 1979 for Allan Cormack. Cormack had sabbaticals at Harvard and probably has done part of his work on tomography here. Tomography helps today for example for cancer treatment.



MATRIX ALGEBRA. Matrices can be added, subtracted if they have the same size:

$$A+B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

They can also be scaled by a scalar λ :

$$\lambda A = \lambda \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$

TRANSFORMATIONS. A **transformation** T from a set X to a set Y is a rule, which assigns to every element in X an element $y = T(x)$ in Y . One calls X the domain and Y the codomain. A transformation is also called a **map**.

LINEAR TRANSFORMATION. A map T from \mathbf{R}^n to \mathbf{R}^m is called a **linear transformation** if there is a $m \times n$ matrix A such that

$$T(\vec{x}) = A\vec{x}.$$

EXAMPLES.

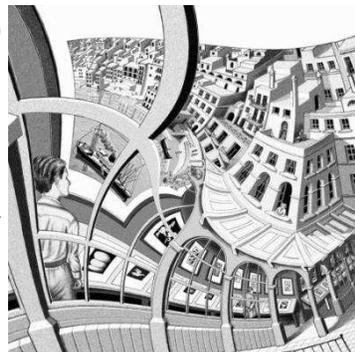
- To the linear transformation $T(x, y) = (3x + 4y, x + 5y)$ belongs the matrix $\begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$. This transformation maps the plane onto itself.
- $T(x) = -3x$ is a linear transformation from the real line onto itself. The matrix is $A = [-3]$.
- To $T(\vec{x}) = \vec{y} \cdot \vec{x}$ from \mathbf{R}^3 to \mathbf{R} belongs the matrix $A = \vec{y} = [y_1 \ y_2 \ y_3]$. This 1×3 matrix is also called a **row vector**. If the codomain is the real axes, one calls the map also a **function**, function defined on space.
- $T(x) = x\vec{y}$ from \mathbf{R} to \mathbf{R}^3 . $A = \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is a 3×1 matrix which is also called a **column vector**. The map defines a line in space.
- $T(x, y, z) = (x, y)$ from \mathbf{R}^3 to \mathbf{R}^2 , A is the 2×3 matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. The map projects space onto a plane.
- To the map $T(x, y) = (x + y, x - y, 2x - 3y)$ belongs the 3×2 matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -3 \end{bmatrix}$. The image of the map is a plane in three dimensional space.
- If $T(\vec{x}) = \vec{x}$, then T is called the **identity transformation**.

PROPERTIES OF LINEAR TRANSFORMATIONS. $T(\vec{0}) = \vec{0}$ $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ $T(\lambda\vec{x}) = \lambda T(\vec{x})$

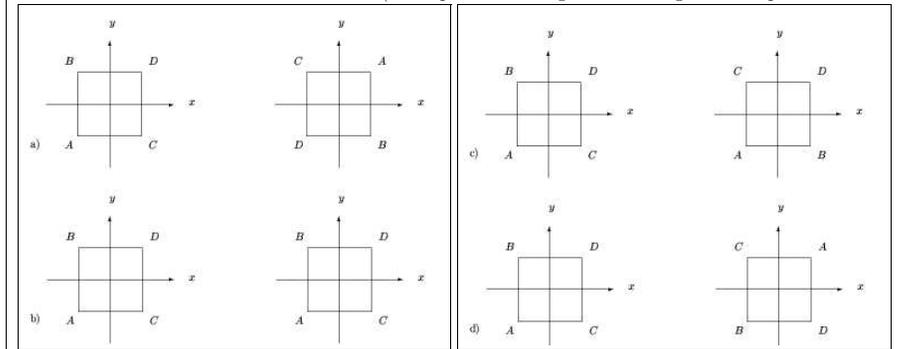
In words: Linear transformations are compatible with addition and scalar multiplication. It does not matter, whether we add two vectors before the transformation or add the transformed vectors.

ON LINEAR TRANSFORMATIONS. Linear transformations generalize the scaling transformation $x \mapsto ax$ in one dimensions. They are important in

- geometry (i.e. rotations, dilations, projections or reflections)
- art (i.e. perspective, coordinate transformations),
- CAD applications (i.e. projections),
- physics (i.e. Lorentz transformations),
- dynamics (linearizations of general maps are linear maps),
- compression (i.e. using Fourier transform or Wavelet transform),
- coding (many codes are linear codes),
- probability (i.e. Markov processes).
- linear equations (inversion is solving the equation)



LINEAR TRANSFORMATION OR NOT? (The square to the right is the image of the square to the left):



COLUMN VECTORS. A linear transformation $T(x) = Ax$ with $A = \begin{bmatrix} | & | & \dots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & \dots & | \end{bmatrix}$ has the property

that the column vector $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are the images of the **standard vectors** $\vec{e}_1 = \begin{bmatrix} 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$, $\vec{e}_i = \begin{bmatrix} 0 \\ \cdot \\ 1 \\ \cdot \end{bmatrix}$, $\vec{e}_n = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$.

In order to find the matrix of a linear transformation, look at the image of the standard vectors and use those to build the columns of the matrix.

QUIZ.

- Find the matrix belonging to the linear transformation, which rotates a cube around the diagonal $(1, 1, 1)$ by 120 degrees ($2\pi/3$).
- Find the linear transformation, which reflects a vector at the line containing the vector $(1, 1, 1)$.

INVERSE OF A TRANSFORMATION. If S is a second transformation such that $S(T\vec{x}) = \vec{x}$, for every \vec{x} , then S is called the **inverse** of T . We will discuss this more later.

SOLVING A LINEAR SYSTEM OF EQUATIONS. $A\vec{x} = \vec{b}$ means to invert the linear transformation $\vec{x} \mapsto A\vec{x}$. If the linear system has exactly one solution, then an inverse exists. We will write $\vec{x} = A^{-1}\vec{b}$ and see that the inverse of a linear transformation is again a linear transformation.

THE BRETSCHER CODE. Otto Bretschers book contains as a motivation a "code", where the encryption happens with the linear map $T(x, y) = (x + 3y, 2x + 5y)$. The map has the inverse $T^{-1}(x, y) = (-5x + 3y, 2x - y)$.



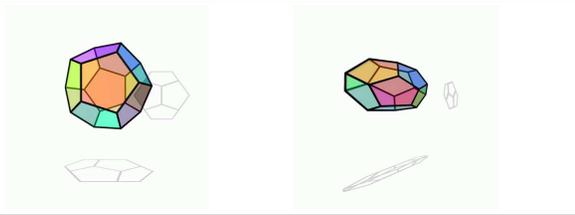
Cryptologists use often the following approach to crack an encryption. If one knows the input and output of some data, one often can decode the key. Assume we know, the enemy uses a Bretscher code and we know that $T(1, 1) = (3, 5)$ and $T(2, 1) = (7, 5)$. How do we get the code? The problem is to find the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

2x2 MATRIX. It is useful to decode the Bretscher code in general. If $ax + by = X$ and $cx + dy = Y$, then $x = (dX - bY)/(ad - bc)$, $y = (cX - aY)/(ad - bc)$. This is a linear transformation with matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

and the corresponding matrix is $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / (ad - bc)$.

"Switch diagonally, negate the wings and scale with a cross".

LINEAR TRANSFORMATIONS DEFORMING A BODY



A CHARACTERIZATION OF LINEAR TRANSFORMATIONS: a transformation T from \mathbf{R}^n to \mathbf{R}^m which satisfies $T(\vec{0}) = \vec{0}$, $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(\lambda\vec{x}) = \lambda T(\vec{x})$ is a linear transformation.
Proof. Call $\vec{v}_i = T(\vec{e}_i)$ and define $S(\vec{x}) = A\vec{x}$. Then $S(\vec{e}_i) = T(\vec{e}_i)$. With $\vec{x} = x_1\vec{e}_1 + \dots + x_n\vec{e}_n$, we have $T(\vec{x}) = T(x_1\vec{e}_1 + \dots + x_n\vec{e}_n) = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$ as well as $S(\vec{x}) = A(x_1\vec{e}_1 + \dots + x_n\vec{e}_n) = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$ proving $T(\vec{x}) = S(\vec{x}) = A\vec{x}$.

SHEAR:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{[Diagram: Shear transformation of a square into a parallelogram]} \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{[Diagram: Shear transformation of a square into a parallelogram]}$$

In general, shears are transformation in the plane with the property that there is a vector \vec{w} such that $T(\vec{w}) = \vec{w}$ and $T(\vec{x}) - \vec{x}$ is a multiple of \vec{w} for all \vec{x} . If \vec{u} is orthogonal to \vec{w} , then $T(\vec{x}) = \vec{x} + (\vec{u} \cdot \vec{x})\vec{w}$.

SCALING:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{[Diagram: Uniform scaling of a square]} \quad A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \text{[Diagram: Uniform scaling of a square]} \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{[Diagram: Anisotropic scaling of a square]}$$

One can also look at transformations which scale x differently then y and where A is a diagonal matrix.

REFLECTION:

$$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix} \quad \text{[Diagram: Reflection across a line]} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{[Diagram: Reflection across the x-axis]}$$

Any reflection at a line has the form of the matrix to the left. A reflection at a line containing a unit vector \vec{u} is $T(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$ with matrix $A = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$

PROJECTION:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{[Diagram: Projection onto the x-axis]} \quad A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{[Diagram: Projection onto the y-axis]}$$

A projection onto a line containing unit vector \vec{u} is $T(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u}$ with matrix $A = \begin{bmatrix} u_1u_1 & u_2u_1 \\ u_1u_2 & u_2u_2 \end{bmatrix}$

ROTATION:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{[Diagram: Reflection through the origin]} \quad A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \quad \text{[Diagram: Rotation of a square]} \quad A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{[Diagram: Rotation-dilation of a square]}$$

Any rotation has the form of the matrix to the right.

ROTATION-DILATION:

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \quad \text{[Diagram: Rotation-dilation of a square]} \quad A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \quad \text{[Diagram: Rotation-dilation of a square]}$$

A rotation dilation is a composition of a rotation by angle $\arctan(y/x)$ and a dilation by a factor $\sqrt{x^2 + y^2}$. If $z = x + iy$ and $w = a + ib$ and $T(x, y) = (X, Y)$, then $X + iY = zw$. So a rotation dilation is tied to the process of the multiplication with a complex number.

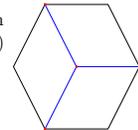
BOOST:

$$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix} \quad \text{[Diagram: Lorentz boost of a square]} \quad \text{The boost is a basic Lorentz transformation in special relativity. It acts on vectors } (x, ct), \text{ where } t \text{ is time, } c \text{ is the speed of light and } x \text{ is space.}$$

Unlike in Galileo transformation $(x, t) \mapsto (x + vt, t)$ (which is a shear), time t also changes during the transformation. The transformation has the effect that it changes length (Lorentz contraction). The angle α is related to v by $\tanh(\alpha) = v/c$. One can write also $A(x, ct) = ((x + vt)/\gamma, t + (v/c^2)/\gamma x)$, with $\gamma = \sqrt{1 - v^2/c^2}$.

ROTATION IN SPACE. Rotations in space are defined by an axes of rotation and an angle. A rotation by 120° around a line containing $(0, 0, 0)$ and $(1, 1, 1)$

belongs to $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ which permutes $\vec{e}_1 \rightarrow \vec{e}_2 \rightarrow \vec{e}_3$.



REFLECTION AT PLANE. To a reflection at the xy -plane belongs the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

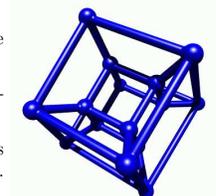
as can be seen by looking at the images of \vec{e}_i . The picture to the right shows the textbook and reflections of it at two different mirrors.



PROJECTION ONTO SPACE. To project a 4d-object into xyz-space, use

$$\text{for example the matrix } A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The picture shows the projection of the four dimensional cube (tesseract, hypercube) with 16 edges $(\pm 1, \pm 1, \pm 1, \pm 1)$. The tesseract is the theme of the horror movie "hypercube".



The Lorentz Boost

Math21b, Oliver Knill

This background information is not part of the course. The relation with special relativity might be fun to know about. We will use the functions $\cosh(x) = (e^x + e^{-x})/2$, $\sinh(x) = (e^x - e^{-x})/2$ on this page.

LORENTZ BOOST. The linear transformation of the plane given by the matrix

$$A = \begin{bmatrix} \cosh(\phi) & \sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{bmatrix}$$

is called the **Lorentz boost**. The transformation $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}$ with $y = ct$ appears in physics.

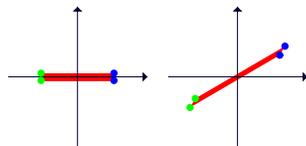
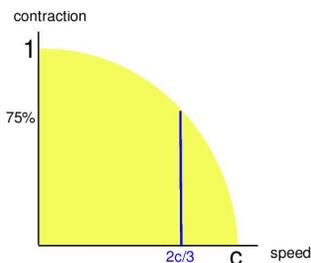
PHYSICAL INTERPRETATIONS. In classical mechanics, when a particle travels with velocity v on the line, its new position satisfies $\tilde{x} = x + tv$.

According to special relativity, this is only an approximation. In reality, the motion is described by $\begin{pmatrix} x \\ ct \end{pmatrix} \mapsto A \begin{pmatrix} x \\ ct \end{pmatrix}$ where A is the above matrix and where the angle ϕ is related to v by the formula $\tanh(\phi) = v/c$. Trigonometric identities give $\sinh(\phi) = (v/c)/\gamma$, $\cosh(\phi) = 1/\gamma$, where $\gamma = \sqrt{1 - v^2/c^2}$. The linear transformation tells then that $A(x, ct) = ((x + vt)/\gamma, t + (v/c^2)/\gamma x)$. For small velocities v , the value of γ is close to 1 and v/c^2 is close to zero so that $A(x, ct)$ is close to $(x + vt, t)$.

LORENTZ CONTRACTION. If we displace a ruler $[a, b]$ with velocity v then its end points are not $[a + tv, b + tv]$ as Newtonian mechanics would tell but $[(a + tv)/\gamma, (b + tv)/\gamma]$. The ruler is by a factor $1/\gamma$ larger, when we see it in a coordinate system where it rests. The constant γ is called the **Lorentz contraction factor**.

For example, for $v = 2/3c$, where c is the speed of light, the contraction is 75 percent. The following picture shows the ruler in the moving coordinate system and in the resting coordinate system.

In the resting coordinate system, the two end points of the ruler have a different time. If a light signal would be sent out simultaneously at the both ends, then this signal would reach the origin at different times. The one to the left earlier than the one to the right. The end point to the left is "younger".)



MAGNETIC FORCE DERIVED FROM ELECTRIC FORCE.

One striking application of the Lorentz transformation is that if you take two wires and let an electric current flow in the same direction, then the distance between the electrons shrinks: the positively charged ions in the wire see a larger electron density than the ion density. The two wires appear negatively "charged" and repel each other. If the currents flow in different directions and we go into a coordinate system, where the electrons are at rest in the first wire, then the ion density of the ions in the same wire appears denser as well as the electron density in the other wire. The two wires then attract each other. The force is proportional to $1/r$.

10/3/2003, THE INVERSE

Math 21b, O. Knill

INVERTIBLE TRANSFORMATIONS. A map T from X to Y is **invertible** if there is for every $y \in Y$ a **unique** point $x \in X$ such that $T(x) = y$.



EXAMPLES.

- 1) $T(x) = x^3$ is invertible from $X = \mathbf{R}$ to $X = Y$.
- 2) $T(x) = x^2$ is not invertible from $X = \mathbf{R}$ to $X = Y$.
- 3) $T(x, y) = (x^2 + 3x - y, x)$ is invertible from $X = \mathbf{R}^2$ to $Y = \mathbf{R}^2$.
- 4) $T(\vec{x}) = A\vec{x}$ linear and $\text{rref}(A)$ has an empty row, then T is not invertible.
- 5) If $T(\vec{x}) = A\vec{x}$ is linear and $\text{rref}(A) = 1_n$, then T is invertible.

INVERSE OF LINEAR TRANSFORMATION. If A is a $n \times n$ matrix and $T: \vec{x} \mapsto A\vec{x}$ has an inverse S , then S is linear. The matrix A^{-1} belonging to $S = T^{-1}$ is called the **inverse matrix** of A .

First proof: check that S is linear using the characterization $S(\vec{a} + \vec{b}) = S(\vec{a}) + S(\vec{b}), S(\lambda\vec{a}) = \lambda S(\vec{a})$ of linearity. Second proof: construct the inverse using Gauss-Jordan elimination.

FINDING THE INVERSE. Let 1_n be the $n \times n$ identity matrix. Start with $[A|1_n]$ and perform Gauss-Jordan elimination. Then

$$\text{rref}([A|1_n]) = [1_n|A^{-1}]$$

Proof. The elimination process actually solves $A\vec{x} = \vec{e}_i$ simultaneously. This leads to solutions \vec{v}_i which are the columns of the inverse matrix A^{-1} because $A^{-1}\vec{e}_i = \vec{v}_i$.

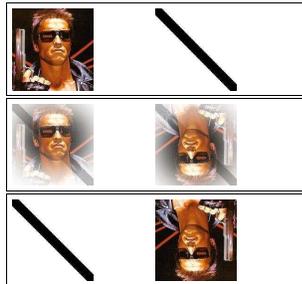
EXAMPLE. Find the inverse of $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} 2 & 6 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array} \right] \quad [A \mid 1_2]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 1 & 4 & 0 & 1 \end{array} \right] \quad [\dots \mid \dots]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1 \end{array} \right] \quad [\dots \mid \dots]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1/2 & 1 \end{array} \right] \quad [1_2 \mid A^{-1}]$$



The inverse is $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1/2 & 1 \end{bmatrix}$.

THE INVERSE OF LINEAR MAPS $R^2 \mapsto R^2$:

If $ad - bc \neq 0$, the inverse of a linear transformation $\vec{x} \mapsto A\vec{x}$ with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by the matrix $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / (ad - bc)$.

SHEAR:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

DIAGONAL:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

REFLECTION:

$$A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

$$A^{-1} = A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

ROTATION:

$$A = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

ROTATION-DILATION:

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} a/r^2 & b/r^2 \\ -b/r^2 & a/r^2 \end{bmatrix}, r^2 = a^2 + b^2$$

BOOST:

$$A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$$

NONINVERTIBLE EXAMPLE. The projection $\vec{x} \mapsto A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a non-invertible transformation.

MORE ON SHEARS. The shears $T(x, y) = (x + ay, y)$ or $T(x, y) = (x, y + ax)$ in \mathbf{R}^2 can be generalized. A shear is a linear transformation which fixes some line L through the origin and which has the property that $T(\vec{x}) - \vec{x}$ is parallel to L for all \vec{x} .

PROBLEM. $T(x, y) = (3x/2 + y/2, y/2 - x/2)$ is a shear along a line L . Find L .

SOLUTION. Solve the system $T(x, y) = (x, y)$. You find that the vector $(1, -1)$ is preserved.

MORE ON PROJECTIONS. A linear map T with the property that $T(T(x)) = T(x)$ is a projection. Examples: $T(\vec{x}) = (\vec{y} \cdot \vec{x})\vec{y}$ is a projection onto a line spanned by a unit vector \vec{y} .

WHERE DO PROJECTIONS APPEAR? CAD: describe 3D objects using projections. A photo of an image is a projection. Compression algorithms like JPG or MPG or MP3 use projections where the high frequencies are cut away.

MORE ON ROTATIONS. A linear map T which preserves the angle between two vectors and the length of each vector is called a **rotation**. Rotations form an important class of transformations and will be treated later in more detail. In two dimensions, every rotation is of the form $x \mapsto A(x)$ with $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$.

An example of a rotations in three dimensions are $\vec{x} \mapsto A\vec{x}$, with $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is a rotation around the z axis.

MORE ON REFLECTIONS. Reflections are linear transformations different from the identity which are equal to their own inverse. Examples:

2D reflections at the origin: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, **2D reflections at a line** $A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$.

3D reflections at origin: $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. **3D reflections at a line** $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. By

the way: in any dimensions, to a reflection at the line containing the unit vector \vec{u} belongs the matrix $[A]_{ij} = 2(u_i u_j) - [1_n]_{ij}$, because $[B]_{ij} = u_i u_j$ is the matrix belonging to the projection onto the line.

The reflection at a line containing the unit vector $\vec{u} = [u_1, u_2, u_3]$ is $A = \begin{bmatrix} u_1^2 - 1 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2^2 - 1 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & u_3^2 - 1 \end{bmatrix}$.

3D reflection at a plane $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Reflections are important symmetries in physics: T (time reflection), P (reflection at a mirror), C (change of charge) are reflections. It seems today that the composition of TCP is a fundamental symmetry in nature.

10/6/2003, MATRIX PRODUCT

Math 21b, O. Knill

MATRIX PRODUCT. If B is a $m \times n$ matrix and A is a $n \times p$ matrix, then BA is defined as the $m \times p$ matrix with entries $(BA)_{ij} = \sum_{k=1}^n B_{ik}A_{kj}$.

EXAMPLE. If B is a 3×4 matrix, and A is a 4×2 matrix then BA is a 3×2 matrix.

$$B = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix}, A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, BA = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 13 \\ 14 & 11 \\ 10 & 5 \end{bmatrix}.$$

COMPOSING LINEAR TRANSFORMATIONS. If $S : \mathbf{R}^m \rightarrow \mathbf{R}^n, x \mapsto Ax$ and $T : \mathbf{R}^n \rightarrow \mathbf{R}^p, y \mapsto By$ are linear transformations, then their composition $T \circ S : x \mapsto B(A(x))$ is a linear transformation from \mathbf{R}^m to \mathbf{R}^p . The corresponding matrix is the matrix product BA .

EXAMPLE. Find the matrix which is a composition of a rotation around the x -axes by an angle $\pi/2$ followed by a rotation around the z -axes by an angle $\pi/2$.

SOLUTION. The first transformation has the property that $e_1 \rightarrow e_1, e_2 \rightarrow e_3, e_3 \rightarrow -e_2$, the second $e_1 \rightarrow e_2, e_2 \rightarrow -e_1, e_3 \rightarrow e_3$. If A is the matrix belonging to the first transformation and B the second, then BA is the matrix to the composition. The composition maps $e_1 \rightarrow -e_2 \rightarrow e_3 \rightarrow e_1$ is a rotation around a long diagonal. $B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$

EXAMPLE. A rotation dilation is the composition of a rotation by $\alpha = \arctan(b/a)$ and a dilation (=scale) by $r = \sqrt{a^2 + b^2}$.

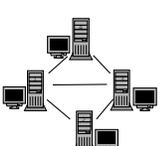
REMARK. Matrix multiplication is a generalization of usual multiplication of numbers or the dot product.

MATRIX ALGEBRA. Note that $AB \neq BA$ in general! Otherwise, the same rules apply as for numbers: $A(BC) = (AB)C, AA^{-1} = A^{-1}A = 1_n, (AB)^{-1} = B^{-1}A^{-1}, A(B+C) = AB + AC, (B+C)A = BA + CA$ etc.

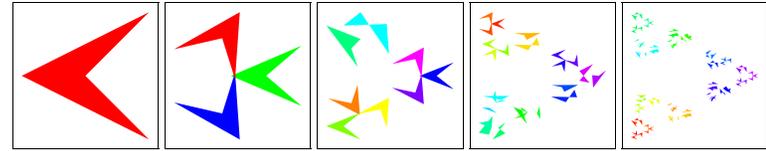
PARTITIONED MATRICES. The entries of matrices can themselves be matrices. If B is a $m \times n$ matrix and A is a $n \times p$ matrix, and assume the entries are $k \times k$ matrices, then BA is a $m \times p$ matrix where each entry $(BA)_{ij} = \sum_{k=1}^n B_{ik}A_{kj}$ is a $k \times k$ matrix. Partitioning matrices can be useful to improve the speed of matrix multiplication (i.e. Strassen algorithm).

EXAMPLE. If $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where A_{ij} are $k \times k$ matrices with the property that A_{11} and A_{22} are invertible, then $B = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$ is the inverse of A .

APPLICATIONS. (The material which follows is for motivation purposes only, more applications appear in the homework).

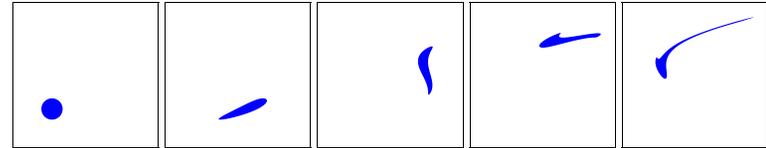



NETWORKS. Let us associate to the computer network (shown at the left) a matrix $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$. To a worm in the first computer we associate a vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. The vector Ax has a 1 at the places, where the worm could be in the next step. The vector $(AA)(x)$ tells, in how many ways the worm can go from the first computer to other hosts in 2 steps. In our case, it can go in three different ways back to the computer itself. Matrices help to solve combinatorial problems (see movie "Good will hunting"). For example, what does $[A^{1000}]_{22}$ tell about the worm infection of the network? What does it mean if A^{100} has no zero entries?



FRACTALS. Closely related to linear maps are **affine maps** $x \mapsto Ax + b$. They are compositions of a linear map with a translation. It is **not** a linear map if $B(0) \neq 0$. Affine maps can be disguised as linear maps in the following way: let $y = \begin{bmatrix} x \\ 1 \end{bmatrix}$ and define the $(n+1) \times (n+1)$ matrix $B = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$. Then $By = \begin{bmatrix} Ax + b \\ 1 \end{bmatrix}$.

Fractals can be constructed by taking for example 3 affine maps R, S, T which contract area. For a given object Y_0 define $Y_1 = R(Y_0) \cup S(Y_0) \cup T(Y_0)$ and recursively $Y_k = R(Y_{k-1}) \cup S(Y_{k-1}) \cup T(Y_{k-1})$. The above picture shows Y_k after some iterations. In the limit, for example if $R(Y_0), S(Y_0)$ and $T(Y_0)$ are disjoint, the sets Y_k converge to a **fractal**, an object with dimension strictly between 1 and 2.



CHAOS. Consider a map in the plane like $T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + 2 \sin(x) - y \\ x \end{bmatrix}$. We apply this map again and again and follow the points $(x_1, y_1) = T(x, y), (x_2, y_2) = T(T(x, y))$, etc. One writes T^n for the n -th iteration of the map and (x_n, y_n) for the image of (x, y) under the map T^n . The linear approximation of the map at a point (x, y) is the matrix $DT(x, y) = \begin{bmatrix} 2 + 2 \cos(x) - 1 & -1 \\ 1 & 0 \end{bmatrix}$. (If $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$, then the row vectors of $DT(x, y)$ are just the gradients of f and g). T is called **chaotic at** (x, y) , if the entries of $D(T^n)(x, y)$ grow exponentially fast with n . By the **chain rule**, $D(T^n)$ is the product of matrices $DT(x_i, y_i)$. For example, T is chaotic at $(0, 0)$. If there is a positive probability to hit a chaotic point, then T is called chaotic.

FALSE COLORS. Any color can be represented as a vector (r, g, b) , where $r \in [0, 1]$ is the red $g \in [0, 1]$ is the green and $b \in [0, 1]$ is the blue component. Changing colors in a picture means applying a transformation on the cube. Let $T : (r, g, b) \mapsto (g, b, r)$ and $S : (r, g, b) \mapsto (r, g, 0)$. What is the composition of these two linear maps?



OPTICS. Matrices help to calculate the motion of light rays through lenses. A light ray $y(s) = x + ms$ in the plane is described by a vector $\begin{bmatrix} x \\ m \end{bmatrix}$. Following the light ray over a distance of length L corresponds to the map $\begin{bmatrix} x \\ m \end{bmatrix} \mapsto \begin{bmatrix} x + mL \\ m \end{bmatrix}$. In the lens, the ray is bent depending on the height x . The transformation in the lens is $(x, m) \mapsto (x, m - kx)$, where k is the strength of the lens.

$$\begin{bmatrix} x \\ m \end{bmatrix} \mapsto A_L \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}, \begin{bmatrix} x \\ m \end{bmatrix} \mapsto B_k \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

Examples:

- 1) Eye looking far: $A_R B_k$.
- 2) Eye looking at distance L : $A_R B_k A_L$.
- 3) Telescope: $B_{k_2} A_L B_{k_1}$. (More about it in problem 80 in section 2.4).

We try out the error correcting code as in the book (problem 53-54 in 3.1).

I) Encoding.

To do so, we encode the letters of the alphabet by pairs of three vectors containing zeros and ones:

- A = (0, 0, 0, 1), (0, 0, 0, 1) B = (0, 0, 0, 1), (0, 0, 1, 0) C = (0, 0, 0, 1), (0, 0, 1, 1)
- D = (0, 0, 0, 1), (0, 1, 0, 1) E = (0, 0, 0, 1), (0, 1, 1, 0) F = (0, 0, 0, 1), (0, 1, 1, 1)
- G = (0, 0, 0, 1), (1, 0, 0, 1) H = (0, 0, 0, 1), (1, 0, 1, 0) I = (0, 0, 0, 1), (1, 0, 1, 1)
- J = (0, 0, 0, 1), (1, 1, 0, 1) K = (0, 0, 0, 1), (1, 1, 1, 0) L = (0, 0, 0, 1), (1, 1, 1, 1)
- M = (0, 0, 1, 0), (0, 0, 0, 1) N = (0, 0, 1, 0), (0, 0, 1, 0) O = (0, 0, 1, 0), (0, 0, 1, 1)
- P = (0, 0, 1, 0), (0, 1, 0, 1) Q = (0, 0, 1, 0), (0, 1, 1, 0) R = (0, 0, 1, 0), (0, 1, 1, 1)
- S = (0, 0, 1, 0), (1, 0, 0, 1) T = (0, 0, 1, 0), (1, 0, 1, 0) U = (0, 0, 1, 0), (1, 0, 1, 1)
- V = (0, 0, 1, 0), (1, 1, 0, 1) W = (0, 0, 1, 0), (1, 1, 1, 0) X = (0, 0, 1, 0), (1, 1, 1, 1)
- Y = (0, 0, 1, 1), (1, 0, 0, 1) Z = (0, 0, 1, 1), (1, 0, 1, 0) ? = (0, 0, 1, 1), (1, 0, 1, 1)
- ! = (0, 0, 1, 1), (1, 0, 0, 1) . = (0, 0, 1, 1), (1, 0, 1, 0) , = (0, 0, 1, 1), (1, 0, 1, 1)

Choose a letter

Look up in the above table the pair (x, y) which belongs to this letter.

$$x = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad y = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Now we build (Mx, My), where M is the matrix $M = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Use 1 + 1 = 0 in the matrix multiplications which follow! Let's go:

FOLD

$$Mx = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$$My = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Now, fold this page at the prescribed fold and show the two vectors Mx, My to your neighbor, who writes down on the top of the second page.

II) Transmission.

You obtain now the vectors Mx, My from your neighbor. Copy the two vectors (Mx, My) of him or her but add one error. To do so, switch one 1 to 0 or one 0 to 1 in the above vectors.

$$u = Mx + e = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \quad v = My + f = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

III) Detect the error e and f.

Detect errors by forming

$$Hu = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

$$Hv = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

Now look in which column Hu or Hv is. Put 0's everywhere in e except at that place, where you put a 1. For example if Hu is the second column, then put a 1 at the second place. We obtain e and f:

$$e = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad f = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

IV) Decode the message.

Let $P \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \\ g \end{bmatrix}$. Determine $Pe = P \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$, $Pf = P \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$. In an error-free transmission (Pu, Pv) would give the right result back. Now

$$Pu = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad Pv = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

satisfy $Pu = x + Pe, Pv = y + Pf$. We recover the original message by subtracting Pe, Pf from that

$$x = Pu - Pe = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad y = Pv - Pf = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

(In the subtraction like addition, use 1 + 1 = 0, -1 - 1 = 0, 1 - 1 = 0, -1 = 1.)

The letter belonging to (x, y) (look it up) is .

10/8/2003, IMAGE AND KERNEL

Math 21b, O. Knill

IMAGE. If $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a linear transformation, then $\{T(\vec{x}) \mid \vec{x} \in \mathbf{R}^n\}$ is called the **image** of T . If $T(\vec{x}) = A\vec{x}$, then the image of T is also called the image of A . We write $\text{im}(A)$ or $\text{im}(T)$.

EXAMPLES.

- 1) If $T(x, y, z) = (x, y, 0)$, then $T(\vec{x}) = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The image of T is the $x - y$ plane.
- 2) If $T(x, y)(\cos(\phi)x - \sin(\phi)y, \sin(\phi)x + \cos(\phi)y)$ is a rotation in the plane, then the image of T is the whole plane.
- 3) If $T(x, y, z) = x + y + z$, then the image of T is \mathbf{R} .

SPAN. The **span** of vectors $\vec{v}_1, \dots, \vec{v}_k$ in \mathbf{R}^n is the set of all combinations $c_1\vec{v}_1 + \dots + c_k\vec{v}_k$, where c_i are real numbers.

PROPERTIES.

The image of a linear transformation $\vec{x} \mapsto A\vec{x}$ is the span of the column vectors of A .
The image of a linear transformation contains 0 and is closed under addition and scalar multiplication.

KERNEL. If $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ is a linear transformation, then the set $\{x \mid T(x) = 0\}$ is called the **kernel** of T . If $T(\vec{x}) = A\vec{x}$, then the kernel of T is also called the kernel of A . We write $\text{ker}(A)$ or $\text{ker}(T)$.

EXAMPLES. (The same examples as above)

- 1) The kernel is the z -axes. Every vector $(0, 0, z)$ is mapped to 0.
- 2) The kernel consists only of the point $(0, 0, 0)$.
- 3) The kernel consists of all vector (x, y, z) for which $x + y + z = 0$. The kernel is a plane.

PROPERTIES.

The kernel of a linear transformation contains 0 and is closed under addition and scalar multiplication.

IMAGE AND KERNEL OF INVERTIBLE MAPS. A linear map $\vec{x} \mapsto A\vec{x}$, $\mathbf{R}^n \mapsto \mathbf{R}^n$ is invertible if and only if $\text{ker}(A) = \{0\}$ if and only if $\text{im}(A) = \mathbf{R}^n$.

HOW DO WE COMPUTE THE IMAGE? The rank of $\text{rref}(A)$ is the dimension of the image. The column vectors of A span the image. (Dimension will be discussed later in detail).

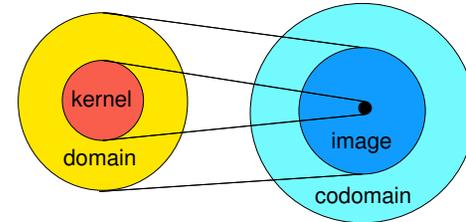
EXAMPLES. (The same examples as above)

- 1) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \cos(\phi) \\ -\sin(\phi) \end{bmatrix}$ and $\begin{bmatrix} \sin(\phi) \\ \cos(\phi) \end{bmatrix}$ 3) The 1D vector $[1]$ spans the image.

HOW DO WE COMPUTE THE KERNEL? Just solve $A\vec{x} = \vec{0}$. Form $\text{rref}(A)$. For every column without leading 1 we can introduce a free variable s_i . If \vec{x} is the solution to $A\vec{x}_i = 0$, where all s_j are zero except $s_i = 1$, then $\vec{x} = \sum_j s_j \vec{x}_j$ is a general vector in the kernel.

EXAMPLE. Find the kernel of the linear map $\mathbf{R}^3 \rightarrow \mathbf{R}^4$, $\vec{x} \mapsto A\vec{x}$ with $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 5 \\ 3 & 9 & 1 \\ -2 & -6 & 0 \end{bmatrix}$. Gauss-Jordan

elimination gives: $B = \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We see one column without leading 1 (the second one). The equation $B\vec{x} = 0$ is equivalent to the system $x + 3y = 0, z = 0$. After fixing $z = 0$, can chose $y = t$ freely and obtain from the first equation $x = -3t$. Therefore, the kernel consists of vectors $t \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. In the book, you have a detailed calculation, in a case, where the kernel is 2 dimensional.



WHY DO WE LOOK AT THE KERNEL?

- It is useful to understand linear maps. To which degree are they non-invertible?
- Helpful to understand quantitatively how many solutions a linear equation $Ax = b$ has. If x is a solution and y is in the kernel of A , then also $A(x + y) = b$, so that $x + y$ solves the system also.

WHY DO WE LOOK AT THE IMAGE?

- A solution $Ax = b$ can be solved if and only if b is in the image of A .
- Knowing about the kernel and the image is useful in the similar way that it is useful to know about the domain and range of a general map and to understand the graph of the map.

In general, the abstraction helps to understand topics like error correcting codes (Problem 53/54 in Bretschers book), where two matrices H, M with the property that $\text{ker}(H) = \text{im}(M)$ appear. The encoding $x \mapsto Mx$ is robust in the sense that adding an error e to the result $Mx \mapsto Mx + e$ can be corrected: $H(Mx + e) = He$ allows to find e and so Mx . This allows to recover $x = PMx$ with a projection P .

PROBLEM. Find $\text{ker}(A)$ and $\text{im}(A)$ for the 1×3 matrix $A = [5, 1, 4]$, a row vector.

ANSWER. $A \cdot \vec{x} = A\vec{x} = 5x + y + 4z = 0$ shows that the kernel is a plane with normal vector $[5, 1, 4]$ through the origin. The image is the codomain, which is \mathbf{R} .

PROBLEM. Find $\text{ker}(A)$ and $\text{im}(A)$ of the linear map $x \mapsto v \times x$, (the cross product with v).

ANSWER. The kernel consists of the line spanned by v , the image is the plane orthogonal to v .

PROBLEM. Fix a vector w in space. Find $\text{ker}(A)$ and image $\text{im}(A)$ of the linear map from \mathbf{R}^6 to \mathbf{R}^3 given by $x, y \mapsto [x, v, y] = (x \times y) \cdot w$.

ANSWER. The kernel consist of all (x, y) such that their cross product orthogonal to w . This means that the plane spanned by x, y contains w .

PROBLEM Find $\text{ker}(T)$ and $\text{im}(T)$ if T is a composition of a rotation R by 90 degrees around the z -axes with with a projection onto the x - z plane.

ANSWER. The kernel of the projection is the y axes. The x axes is rotated into the y axes and therefore the kernel of T . The image is the x - z plane.

PROBLEM. Can the kernel of a square matrix A be trivial if $A^2 = \mathbf{0}$, where $\mathbf{0}$ is the matrix containing only 0?

ANSWER. No: if the kernel were trivial, then A were invertible and A^2 were invertible and be different from $\mathbf{0}$.

PROBLEM. Is it possible that a 3×3 matrix A satisfies $\text{ker}(A) = \mathbf{R}^3$ without $A = \mathbf{0}$?

ANSWER. No, if $A \neq \mathbf{0}$, then A contains a nonzero entry and therefore, a column vector which is nonzero.

PROBLEM. What is the kernel and image of a projection onto the plane $\Sigma : x - y + 2z = 0$?

ANSWER. The kernel consists of all vectors orthogonal to Σ , the image is the plane Σ .

PROBLEM. Given two square matrices A, B and assume $AB = BA$. You know $\text{ker}(A)$ and $\text{ker}(B)$. What can you say about $\text{ker}(AB)$?

ANSWER. $\text{ker}(A)$ is contained in $\text{ker}(BA)$. Similar $\text{ker}(B)$ is contained in $\text{ker}(AB)$. Because $AB = BA$, the kernel of AB contains both $\text{ker}(A)$ and $\text{ker}(B)$. (It can be bigger: $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.)

PROBLEM. What is the kernel of the partitioned matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ if $\text{ker}(A)$ and $\text{ker}(B)$ are known?

ANSWER. The kernel consists of all vectors (\vec{x}, \vec{y}) , where $\vec{x} \in \text{ker}(A)$ and $\vec{y} \in \text{ker}(B)$.

LINEAR SUBSPACE. A subset X of \mathbf{R}^n which is closed under addition and scalar multiplication is called a **linear subspace** of \mathbf{R}^n .

WHICH OF THE FOLLOWING SETS ARE LINEAR SPACES?

- a) The kernel of a linear map.
- b) The image of a linear map.
- c) The upper half plane.
- d) the line $x + y = 0$.
- e) The plane $x + y + z = 1$.
- f) The unit circle.

BASIS. A set of vectors $\vec{v}_1, \dots, \vec{v}_m$ is a **basis** of a linear subspace X of \mathbf{R}^n if they are **linear independent** and if they **span** the space X . Linear independent means that there are no nontrivial **linear relations** $a_1\vec{v}_1 + \dots + a_m\vec{v}_m = 0$. Spanning the space means that every vector \vec{v} can be written as a linear combination $\vec{v} = a_1\vec{v}_1 + \dots + a_m\vec{v}_m$ of basis vectors. A **linear subspace** is a set containing $\{\vec{0}\}$ which is closed under addition and scaling.



EXAMPLE 1) The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis in the three dimensional space.

If $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$, then $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$ and this representation is unique. We can find the coefficients by solving

$A\vec{x} = \vec{v}$, where A has the v_i as column vectors. In our case, $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$ had the unique solution $x = 1, y = 2, z = 3$ leading to $\vec{v} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$.

- EXAMPLE 2) Two nonzero vectors in the plane which are not parallel form a basis.
- EXAMPLE 3) Three vectors in \mathbf{R}^3 which are in a plane form **not a basis**.
- EXAMPLE 4) Two vectors in \mathbf{R}^3 do **not** form a basis.
- EXAMPLE 5) Three nonzero vectors in \mathbf{R}^3 which are not contained in a single plane form a basis in \mathbf{R}^3 .
- EXAMPLE 6) The columns of an invertible $n \times n$ matrix form a basis in \mathbf{R}^n as we will see.

FACT. If $\vec{v}_1, \dots, \vec{v}_n$ is a basis, then every vector \vec{v} can be represented **uniquely** as a linear combination of the \vec{v}_i .
 $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.

REASON. There is at least one representation because the vectors \vec{v}_i span the space. If there were two different representations $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$ and $\vec{v} = b_1\vec{v}_1 + \dots + b_n\vec{v}_n$, then subtraction would lead to $0 = (a_1 - b_1)\vec{v}_1 + \dots + (a_n - b_n)\vec{v}_n$. This nontrivial linear relation of the v_i is forbidden by assumption.

FACT. If n vectors $\vec{v}_1, \dots, \vec{v}_n$ span a space and $\vec{w}_1, \dots, \vec{w}_m$ are linear independent, then $m \leq n$.
 REASON. This is intuitively clear in dimensions up to 3. You can not have more than 4 vectors in space which are linearly independent. We will give a precise reason later.

A BASIS DEFINES AN INVERTIBLE MATRIX. The $n \times n$ matrix $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & & | \end{bmatrix}$ is invertible if and only if $\vec{v}_1, \dots, \vec{v}_n$ define a basis in \mathbf{R}^n .

EXAMPLE. In the example 1), the 3×3 matrix A is invertible.

FINDING A BASIS FOR THE KERNEL. To solve $Ax = 0$, we bring the matrix A into the reduced row echelon form $\text{rref}(A)$. For every non-leading entry in $\text{rref}(A)$, we will get a free variable t_i . Writing the system $Ax = 0$ with these free variables gives us an equation $\vec{x} = \sum_i t_i \vec{v}_i$. The vectors \vec{v}_i form a basis of the kernel of A .

REMARK. The problem to find a basis for all vectors \vec{w}_i which are orthogonal to a given set of vectors, is equivalent to the problem to find a basis for the kernel of the matrix which has the vectors \vec{w}_i in its rows.

FINDING A BASIS FOR THE IMAGE. Bring the $m \times n$ matrix A into the form $\text{rref}(A)$. Call a column a **pivot column**, if it contains a leading 1. The corresponding set of column vectors of the original matrix A form a basis for the image because they are linearly independent and are in the image. Assume there are k of them. They span the image because there are $(k - n)$ non-leading entries in the matrix.

REMARK. The problem to find a basis of the subspace generated by $\vec{v}_1, \dots, \vec{v}_n$, is the problem to find a basis for the image of the matrix A with column vectors $\vec{v}_1, \dots, \vec{v}_n$.

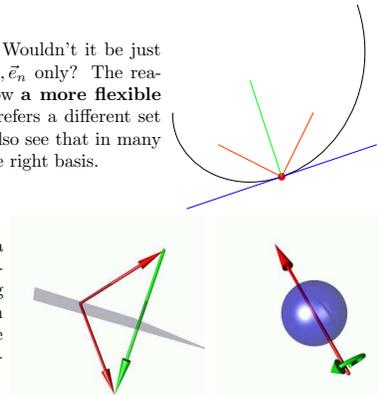
- EXAMPLES.
- 1) Two vectors on a line are linear dependent. One is a multiple of the other.
 - 2) Three vectors in the plane are linear dependent. One can find a relation $a\vec{v}_1 + b\vec{v}_2 = \vec{v}_3$ by changing the size of the lengths of the vectors \vec{v}_1, \vec{v}_2 until \vec{v}_3 becomes the diagonal of the parallelogram spanned by \vec{v}_1, \vec{v}_2 .
 - 3) Four vectors in three dimensional space are linearly dependent. As in the plane one can change the length of the vectors to make \vec{v}_4 a diagonal of the parallelepiped spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

EXAMPLE. Let A be the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. In reduced row echelon form is $B = \text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

To determine a basis of the kernel we write $Bx = 0$ as a system of linear equations: $x + y = 0, z = 0$. The variable y is the free variable. With $y = t, x = -t$ is fixed. The linear system $\text{rref}(A)x = 0$ is solved by $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. So, $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis of the kernel.

EXAMPLE. Because the first and third vectors in $\text{rref}(A)$ are columns with leading 1's, the first and third columns $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ of A form a basis of the image of A .

WHY DO WE INTRODUCE BASIS VECTORS? Wouldn't it be just easier to look at the standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$ only? The reason for more general basis vectors is that they allow a **more flexible adaptation** to the situation. A person in Paris prefers a different set of basis vectors than a person in Boston. We will also see that in many applications, problems can be solved easier with the right basis.



For example, to describe the reflection of a ray at a plane or at a curve, it is preferable to use basis vectors which are tangent or orthogonal. When looking at a rotation, it is good to have one basis vector in the axis of rotation, the other two orthogonal to the axis. Choosing the right basis will be especially important when studying differential equations.

A PROBLEM. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Find a basis for $\ker(A)$ and $\text{im}(A)$.

SOLUTION. From $\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ we see that $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is in the kernel. The two column vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, [2, 1, 1]$ of A form a basis of the image because the first and third column are pivot columns.

REVIEW LINEAR SPACE. X **linear space**: $\vec{0} \in X$ closed under addition and scalar multiplication.

Examples: $\mathbf{R}^n, X = \ker(A), X = \text{im}(A)$ are linear spaces.

REVIEW BASIS. $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\} \subset X$
 \mathcal{B} linear independent: $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = 0$ implies $c_1 = \dots = c_n = 0$.

\mathcal{B} span X : $\vec{v} \in X$ then $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.
 \mathcal{B} basis: both linear independent and span.



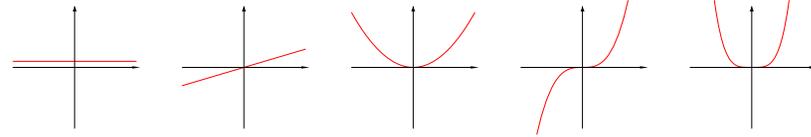
BASIS: ENOUGH BUT NOT TOO MUCH. The spanning condition for a basis assures that there are **enough** vectors to represent any other vector, the linear independence condition assures that there are **not too many** vectors. A basis is, where J.Lo meets A.Hi: Left: J.Lopez in "Enough", right "The man who new too much" by A.Hitchcock



AN UNUSUAL EXAMPLE. Let X be the space of polynomials up to degree 4. For example $p(x) = 3x^4 + 2x^3 + x + 5$ is an element in this space. It is straightforward to check that X is a linear space. The "zero vector" is the function $f(x) = 0$ which is zero everywhere. We claim that $e_1(x) = 1, e_2(x) = x, e_3(x) = x^2, e_4(x) = x^3$ and $e_5(x) = x^4$ form a basis in X .

PROOF. The vectors span the space: every polynomial $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$ is a sum $f = c_0e_1 + c_1e_2 + c_2e_3 + c_3e_4 + c_4e_5$ of basis elements.

The vectors are linearly independent: a nontrivial relation $0 = c_0e_1 + c_1e_2 + c_2e_3 + c_3e_4 + c_4e_5$ would mean that $c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 = 0$ for all x which is not possible unless all c_j are zero.



DIMENSION. The number of elements in a basis of X is independent of the choice of the basis. It is called the **dimension** of X .

UNIQUE REPRESENTATION. $\vec{v}_1, \dots, \vec{v}_n \in X$ **basis** \Rightarrow every $\vec{v} \in X$ can be written uniquely as a sum $\vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$.

EXAMPLES. The dimension of $\{0\}$ is zero. The dimension of any line is 1. The dimension of a plane is 2, the dimension of three dimensional space is 3. The dimension is independent on where the space is embedded in. For example: a line in the plane and a line in space have dimension 1.

IN THE UNUSUAL EXAMPLE. The set of polynomials of degree ≤ 4 form a linear space of dimension 5.

REVIEW: KERNEL AND IMAGE. We can construct a basis of the kernel and image of a linear transformation $T(x) = Ax$ by forming $B = \text{rref}A$. The set of Pivot columns in A form a basis of the image of T , a basis for the kernel is obtained by solving $Bx = 0$ and introducing free variables for each non-pivot column.

EXAMPLE. Let X the linear space from above. Define the linear transformation $T(f)(x) = f'(x)$. For example: $T(x^3 + 2x^4) = 3x^2 + 8x^3$. Find a basis for the kernel and image of this transformation.

SOLUTION. Because $T(e_1) = 0, T(e_2) = e_1, T(e_3) = 2e_2, T(e_4) = 3e_3, T(e_5) = 4e_4$, the matrix of T is $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which is almost in row reduced echelon form. You see that the last four columns are pivot columns. The kernel is spanned by e_1 which corresponds to the constant function $f(x) = 1$. The image is the 4 dimensional space of polynomials of degree ≤ 3 .

Mathematicians call a fact a "lemma" if it is used to prove a theorem and if does not deserve the be honored by the name "theorem":

LEMMA. If n vectors $\vec{v}_1, \dots, \vec{v}_n$ span a space and $\vec{w}_1, \dots, \vec{w}_m$ are linearly independent, then $m \leq n$.

REASON. Assume $m > n$. Because \vec{v}_i span, each vector \vec{w}_i can be written as $\sum_j a_{ij}\vec{v}_j = \vec{w}_i$. After doing

Gauss-Jordan elimination of the augmented $(m \times (n+1))$ -matrix $\left[\begin{array}{ccc|ccc} a_{11} & \dots & a_{1n} & \dots & \dots & \vec{w}_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} & \dots & \dots & \vec{w}_m \end{array} \right]$ which belong to

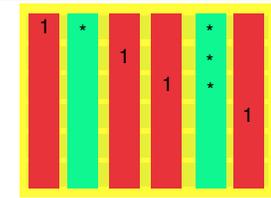
these relations, we end up with a matrix, which has in the last line $\left[0 \quad \dots \quad 0 \quad \dots \quad 0 \quad \left| \quad b_1\vec{w}_1 + \dots + b_m\vec{w}_m \right. \right]$. But this means that $b_1\vec{w}_1 + \dots + b_m\vec{w}_m = 0$ and this is a nontrivial relation between the \vec{w}_i . Contradiction. The assumption $m > n$ was absurd.

THEOREM. Given a basis $\mathcal{A} = \{v_1, \dots, v_n\}$ and a basis $\mathcal{B} = \{w_1, \dots, w_m\}$ of X , then $m = n$.

PROOF. Because \mathcal{A} spans X and \mathcal{B} is linearly independent, we know that $n \leq m$. Because \mathcal{B} spans X and \mathcal{A} is linearly independent also $m \leq n$ holds. Together, $n \leq m$ and $m \leq n$ implies $n = m$.

DIMENSION OF THE KERNEL. The number of columns in $\text{rref}(A)$ without leading 1's is the **dimension of the kernel** $\dim(\ker(A))$: we can introduce a parameter for each such column when solving $Ax = 0$ using Gauss-Jordan elimination.

DIMENSION OF THE IMAGE. The number of **leading 1** in $\text{rref}(A)$, the rank of A is the **dimension of the image** $\dim(\text{im}(A))$ because every such leading 1 produces a different column vector (called **pivot column vectors**) and these column vectors are linearly independent.



DIMENSION FORMULA: $(A : \mathbf{R}^n \rightarrow \mathbf{R}^m)$

$$\dim(\ker(A)) + \dim(\text{im}(A)) = n$$

EXAMPLE: A invertible is equivalent that the dimension of the image is n and that the $\dim(\ker(A)) = 0$.

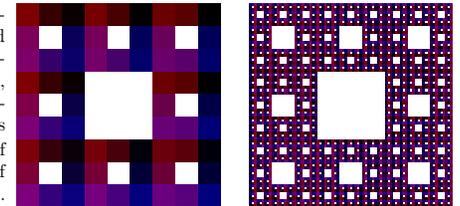
PROOF. There are n columns. $\dim(\ker(A))$ is the number of columns without leading 1, $\dim(\text{im}(A))$ is the number of columns with leading 1.

EXAMPLE. In the space X of polynomials of degree 4 define $T(f)(x) = f''(x)$. The kernel consists of linear polynomials spanned by e_1, e_2 , the image consists of all polynomials of degree ≤ 2 . It is spanned by e_3, e_4, e_5 . Indeed $\dim(\ker(T)) + \dim(\text{im}(T)) = 2 + 3 = 5 = n$.

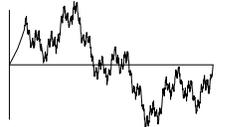
FRactal Dimension. Mathematicians study objects with non-integer dimension since the early 20'th century. The topic became fashion in the 80'ies, when people started to generate fractals on computers. To define fractals, the notion of dimension is extended: define a **s-volume of accuracy r** of a bounded set X in \mathbf{R}^n as the infimum of all $h_{s,r}(X) = \sum U_j |U_j|^s$, where U_j are cubes of length $\leq r$ covering X and $|U_j|$ is the length of U_j . The **s-volume** is then defined as the limit $h_s(X)$ of $h_{s,r}(X) = h_{s,r}(X)$ when $r \rightarrow 0$. The **dimension** is the limiting value s , where $h_s(X)$ jumps from 0 to ∞ . Examples:

- 1) A smooth curve X of length 1 in the plane can be covered with n squares U_j of length $|U_j| = 1/n$ and $h_{s,1/n}(X) = \sum_{j=1}^n (1/n)^s = n(1/n)^s$. If $s < 1$, this converges, if $s > 1$ it diverges for $n \rightarrow \infty$. So $\dim(X) = 1$.
- 2) A square X in space of area 1 can be covered with n^2 cubes U_j of length $|U_j| = 1/n$ and $h_{s,1/n}(X) = \sum_{j=1}^{n^2} (1/n)^s = n^2(1/n)^s$ which converges to 0 for $s < 2$ and diverges for $s > 2$ so that $\dim(X) = 2$.

- 3) The **Shirpinski carpet** is constructed recursively by dividing a square in 9 equal squares and cutting away the middle one, repeating this procedure with each of the squares etc. At the k 'th step, we need 8^k squares of length $1/3^k$ to cover the carpet. The s -volume $h_{s,1/3^k}(X)$ of accuracy $1/3^k$ is $8^k (1/3^k)^s = 8^k / 3^{ks}$, which goes to 0 for $k \rightarrow \infty$ if $3^{ks} < 8^k$ or $s < d = \log(8)/\log(3)$ and diverges if $s > d$. The dimension is $d = \log(8)/\log(3) = 1.893$.



INFINITE DIMENSIONS. Linear spaces also can have infinite dimensions. An example is the set X of all continuous maps from the real \mathbf{R} to \mathbf{R} . It contains all polynomials and because X_n the space of polynomials of degree n with dimension $n + 1$ is contained in X , the space X is infinite dimensional. By the way, there are functions like $g(x) = \sum_{n=0}^{\infty} \sin(2^n x) / 2^n$ in X which have graphs of fractal dimension > 1 and which are not differentiable at any point x .



B-COORDINATES. Given a basis $\vec{v}_1, \dots, \vec{v}_n$, define the matrix $S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}$. It is invertible. If $\vec{x} = \sum_i c_i \vec{v}_i$, then c_i are called the **B-coordinates** of \vec{v} . We write $[\vec{x}]_B = \begin{bmatrix} c_1 \\ \dots \\ c_n \end{bmatrix}$. If $\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$, we have $\vec{x} = S([\vec{x}]_B)$.

B-coordinates of \vec{x} are obtained by applying S^{-1} to the coordinates of the standard basis:
 $[\vec{x}]_B = S^{-1}(\vec{x})$.

EXAMPLE. If $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, then $S = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$. A vector $\vec{v} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$ has the coordinates

$$S^{-1}\vec{v} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

Indeed, as we can check, $-3\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

EXAMPLE. Let V be the plane $x + y - z = 1$. Find a basis, in which every vector in the plane has the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$. SOLUTION. Find a basis, such that two vectors v_1, v_2 are in the plane and such that a third vector v_3 is linearly independent to the first two. Since $(1, 0, 1), (0, 1, 1)$ are points in the plane and $(0, 0, 0)$ is in the plane, we can choose $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ which is perpendicular to the plane.

EXAMPLE. Find the coordinates of $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ with respect to the basis $B = \{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.

We have $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Therefore $[v]_B = S^{-1}\vec{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Indeed $-1\vec{v}_1 + 3\vec{v}_2 = \vec{v}$.

B-MATRIX. If $B = \{v_1, \dots, v_n\}$ is a basis in \mathbf{R}^n and T is a linear transformation on \mathbf{R}^n , then the **B-matrix** of T is defined as

$$B = \begin{bmatrix} | & \dots & | \\ [T(\vec{v}_1)]_B & \dots & [T(\vec{v}_n)]_B \\ | & \dots & | \end{bmatrix}$$

COORDINATES HISTORY. Cartesian geometry was introduced by Fermat and Descartes (1596-1650) around 1636. It had a large influence on mathematics. Algebraic methods were introduced into geometry. The beginning of the vector concept came only later at the beginning of the 19th Century with the work of Bolzano (1781-1848). The full power of coordinates becomes possible if we allow to chose our coordinate system adapted to the situation. Descartes biography shows how far dedication to the teaching of mathematics can go ...:

(...) In 1649 Queen Christina of Sweden persuaded Descartes to go to Stockholm. However the Queen wanted to draw tangents at 5 a.m. in the morning and Descartes broke the habit of his lifetime of getting up at 11 o'clock. After only a few months in the cold northern climate, walking to the palace at 5 o'clock every morning, he died of pneumonia.



Fermat



Descartes



Christina



Bolzano

CREATIVITY THROUGH LAZINESS? Legend tells that Descartes (1596-1650) introduced coordinates while lying on the bed, watching a fly (around 1630), that Archimedes (285-212 BC) discovered a method to find the volume of bodies while relaxing in the bath and that Newton (1643-1727) discovered Newton's law while lying under an apple tree. Other examples are August Kekulé's analysis of the Benzene molecular structure in a dream (a snake biting in its tail revealed the ring structure) or Steven Hawking's discovery that black holes can radiate (while shaving). While unclear which of this is actually true, there is a pattern:



According David Perkins (at Harvard school of education): "The Eureka effect", many creative breakthroughs have in common: a **long search** without apparent progress, a prevailing moment and **break through**, and finally, a transformation and **realization**. A breakthrough in a lazy moment is typical - but only after long struggle and hard work.

EXAMPLE. Let T be the reflection at the plane $x + 2y + 3z = 0$. Find the transformation matrix B in the basis $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$. Because $T(\vec{v}_1) = -\vec{v}_1 = [-\vec{e}_1]_B, T(\vec{v}_2) = \vec{v}_2 = [\vec{e}_2]_B, T(\vec{v}_3) = -\vec{v}_3 = -[\vec{e}_3]_B$, the solution is $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

SIMILARITY. The **B** matrix of A is $B = S^{-1}AS$, where $S = \begin{bmatrix} | & \dots & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & \dots & | \end{bmatrix}$. One says B is **similar** to A .

EXAMPLE. If A is similar to B , then $A^2 + A + 1$ is similar to $B^2 + B + 1$. $B = S^{-1}AS, B^2 = S^{-1}B^2S, S^{-1}S = 1, S^{-1}(A^2 + A + 1)S = B^2 + B + 1$.

PROPERTIES OF SIMILARITY. A, B similar and B, C similar, then A, C are similar. If A is similar to B , then B is similar to A .

QUIZZ: If A is a 2×2 matrix and let $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, What is $S^{-1}AS$?

MAIN IDEA OF CONJUGATION S. The transformation S^{-1} maps the coordinates from the standard basis into the coordinates of the new basis. In order to see what a transformation A does in the new coordinates, we map it back to the old coordinates, apply A and then map it back again to the new coordinates: $B = S^{-1}AS$.

The transformation in standard coordinates. $\vec{v} \xrightarrow{A} A\vec{v}$ The transformation in **B**-coordinates. $\vec{w} = [\vec{v}]_B \xrightarrow{B} B\vec{w}$

QUESTION. Can the matrix A which belongs to a projection from \mathbf{R}^3 to a plane $x + y + 6z = 0$ be similar to a matrix which is a rotation by 20 degrees around the z axis? No: a non-invertible A can not be similar to an invertible B : if it were, the inverse $A = SBS^{-1}$ would exist: $A^{-1} = SB^{-1}S^{-1}$.

PROBLEM. Find a clever basis for the reflection of a light ray at the line $x + 2y = 0$. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

SOLUTION. You can achieve $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ with $S = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

PROBLEM. Are all shears $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ with $a \neq 0$ similar? Yes, use a basis $\vec{v}_1 = a\vec{e}_1$ and $\vec{v}_2 = \vec{e}_2$.

PROBLEM. You know $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ with $S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. Find $e^A = 1 + A + A^2 + A^3/3! + \dots$ SOLUTION. Because $B^k = S^{-1}A^kS$ for every k we have $e^A = Se^BS^{-1} = \begin{bmatrix} 1/e & 0 \\ e + 1/e & e \end{bmatrix}$.

10/20/2003, ORTHOGONALITY

Math 21b, O. Knill

ORTHOGONALITY. \vec{v} and \vec{w} are called **orthogonal** if $\vec{v} \cdot \vec{w} = 0$.

Examples. 1) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 6 \\ -3 \end{bmatrix}$ are orthogonal in \mathbf{R}^2 . 2) \vec{v} and w are both orthogonal to $\vec{v} \times \vec{w}$ in \mathbf{R}^3 .

\vec{v} is called a **unit vector** if $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = 1$. $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ are called **orthogonal** if they are pairwise orthogonal. They are called **orthonormal** if they are also unit vectors. A basis is called an **orthonormal basis** if it is orthonormal. For an orthonormal basis, the matrix $A_{ij} = \vec{v}_i \cdot \vec{v}_j$ is the unit matrix.

FACT. Orthogonal vectors are linearly independent and n orthogonal vectors in \mathbf{R}^n form a basis.

Proof. The dot product of a **linear relation** $a_1\vec{v}_1 + \dots + a_n\vec{v}_n = 0$ with \vec{v}_k gives $a_k\vec{v}_k \cdot \vec{v}_k = a_k\|\vec{v}_k\|^2 = 0$ so that $a_k = 0$. If we have n linear independent vectors in \mathbf{R}^n then they automatically span the space.

ORTHOGONAL COMPLEMENT. A vector $\vec{w} \in \mathbf{R}^n$ is called **orthogonal** to a linear space V if \vec{w} is orthogonal to every vector in $\vec{v} \in V$. The **orthogonal complement** of a linear space V is the set W of all vectors which are orthogonal to V . It forms a linear space because $\vec{v} \cdot \vec{w}_1 = 0, \vec{v} \cdot \vec{w}_2 = 0$ implies $\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = 0$.

ORTHOGONAL PROJECTION. The **orthogonal projection** onto a linear space V with **orthonormal** basis $\vec{v}_1, \dots, \vec{v}_n$ is the linear map $T(\vec{x}) = \text{proj}_V(x) = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \dots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$. The vector $\vec{x} - \text{proj}_V(\vec{x})$ is in the orthogonal complement of V . (Note that \vec{v}_i in the projection formula are unit vectors, they have also to be orthogonal.)

SPECIAL CASE. For an orthonormal basis \vec{v}_i , one can write $\vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \dots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$.

PYTHAGORAS: If \vec{x} and \vec{y} are orthogonal, then $\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$. Proof. Expand $(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y})$. PROJECTIONS DO NOT INCREASE LENGTH: $\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$. Proof. Use Pythagoras: on $\vec{x} = \text{proj}_V(\vec{x}) + (\vec{x} - \text{proj}_V(\vec{x}))$. If $\|\text{proj}_V(\vec{x})\| = \|\vec{x}\|$, then \vec{x} is in V .

CAUCHY-SCHWARTZ INEQUALITY: $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$. Proof: $\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos(\alpha)$.

If $|\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\|$, then \vec{x} and \vec{y} are parallel.

TRIANGLE INEQUALITY: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$. Proof: $(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\vec{x} \cdot \vec{y} \leq \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\| \|\vec{y}\| = (\|\vec{x}\| + \|\vec{y}\|)^2$.

ANGLE. The **angle** between two vectors \vec{x}, \vec{y} is

$$\alpha = \arccos \left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} \right).$$



CORRELATION. $\cos(\alpha) = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$ is called the **correlation** between \vec{x} and \vec{y} . It is a number in $[-1, 1]$.



EXAMPLE. The angle between two orthogonal vectors is 90 degrees or 270 degrees. If \vec{x} and \vec{y} represent data showing the deviation from the mean, then $\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$ is called the **statistical correlation** of the data.

QUESTION. Express the fact that \vec{x} is in the kernel of a matrix A using orthogonality.

ANSWER: $A\vec{x} = 0$ means that $\vec{w}_k \cdot \vec{x} = 0$ for every row vector \vec{w}_k of \mathbf{R}^n .

REMARK. We will call later the matrix A^T , obtained by switching rows and columns of A the **transpose** of A . You see already that the image of A^T is orthogonal to the kernel of A .

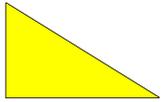
QUESTION. Find a basis for the orthogonal complement of the linear space V spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}$.

ANSWER: The orthogonality of $\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}$ to the two vectors means solving the linear system of equations $x +$

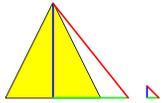
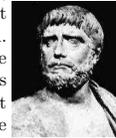
$2y + 3z + 4u = 0, 4x + 5y + 6z + 7u = 0$. An other way to solve it: the kernel of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ is the orthogonal complement of V . This reduces the problem to an older problem.

ON THE RELEVANCE OF ORTHOGONALITY.

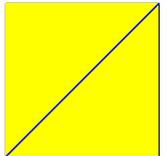
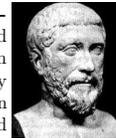
1) During the pyramid age in Egypt (from -2800 til -2300 BC), the Egyptians used ropes divided into length ratios 3 : 4 : 5 to build triangles. This allowed them to triangulate areas quite precisely: for example to build irrigation canals because the Nile was reshaping the land constantly or to build the pyramids: for the **great pyramid at Giza** with a base length of 230 meters, the average error on each side is less then 20cm, an error of less then 1/1000. A key to achieve this was **orthogonality**.



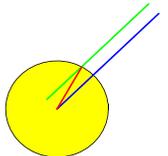
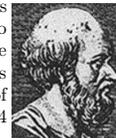
2) During one of Thales (-624 til -548 BC) journeys to Egypt, he used a geometrical trick to **measure the height** of the great pyramid. He measured the size of the shadow of the pyramid. Using a stick, he found the relation between the length of the stick and the length of its shadow. The same length ratio applies to the pyramid (**orthogonal** triangles). Thales found also that triangles inscribed into a circle and having as the base as the diameter must have a right angle.



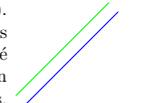
3) The Pythagoreans (-572 until -507) were interested in the discovery that the squares of a lengths of a triangle with two **orthogonal** sides would add up as $a^2 + b^2 = c^2$. They were puzzled in assigning a length to the diagonal of the unit square, which is $\sqrt{2}$. This number is irrational because $\sqrt{2} = p/q$ would imply that $q^2 = 2p^2$. While the prime factorization of q^2 contains an even power of 2, the prime factorization of $2p^2$ contains an odd power of 2.



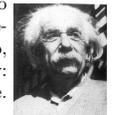
4) Eratosthenes (-274 until 194) realized that while the sun rays were **orthogonal** to the ground in the town of Scene, this did no more do so at the town of Alexandria, where they would hit the ground at 7.2 degrees). Because the distance was about 500 miles and 7.2 is 1/50 of 360 degrees, he measured the circumference of the earth as 25'000 miles - pretty close to the actual value 24'874 miles.



5) Closely related to **orthogonality** is **parallelism**. For a long time mathematicians tried to prove Euclid's parallel axiom using other postulates of Euclid (-325 until -265). These attempts had to fail because there are geometries in which parallel lines always meet (like on the sphere) or geometries, where parallel lines never meet (the Poincaré half plane). Also these geometries can be studied using linear algebra. The geometry on the sphere with **rotations**, the geometry on the half plane uses Möbius transformations, 2×2 matrices with determinant one.



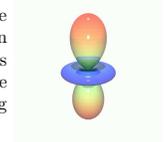
6) The question whether the angles of a right triangle are in reality always add up to 180 degrees became an issue when geometries were discovered, in which the measurement depends on the position in space. Riemannian geometry, founded 150 years ago, is the foundation of **general relativity** a theory which describes gravity geometrically: the presence of mass bends space-time, where the dot product can depend on space. **Orthogonality** becomes relative too.



7) In **probability theory** the notion of **independence** or **decorrelation** is used. For example, when throwing a dice, the number shown by the first dice is independent and decorrelated from the number shown by the second dice. Decorrelation is identical to **orthogonality**, when vectors are associated to the random variables. The **correlation coefficient** between two vectors \vec{v}, \vec{w} is defined as $\vec{v} \cdot \vec{w} / (\|\vec{v}\| \|\vec{w}\|)$. It is the cosine of the angle between these vectors.



8) In **quantum mechanics**, states of atoms are described by functions in a linear space of functions. The states with energy $-E_B/n^2$ (where $E_B = 13.6eV$ is the Bohr energy) in a hydrogen atom. States in an atom are **orthogonal**. Two states of two different atoms which don't interact are **orthogonal**. One of the challenges in quantum computing, where the computation deals with qubits (=vectors) is that orthogonality is not preserved during the computation. Different states can interact. This coupling is called **decoherence**.



DEFINITIONS.

- Linear space** $\vec{0} \in X, \vec{x}, \vec{y} \in X, \lambda \in \mathbf{R} \Rightarrow \vec{x} + \vec{y} \in X, \lambda \vec{x} \in X$.
- Matrix** A is a $m \times n$ matrix, it has n columns and m rows, maps \mathbf{R}^n to \mathbf{R}^m .
- Square matrix** $n \times n$ matrix, maps \mathbf{R}^n to \mathbf{R}^n .
- Vector** $n \times 1$ matrix = column vector, $1 \times n$ matrix = row vector.
- Linear transformation** $T: \mathbf{R}^n \rightarrow \mathbf{R}^m, \vec{x} \mapsto A\vec{x}, T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}), T(\lambda\vec{x}) = \lambda T(\vec{x})$.
- Column vector** of A are images of standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$.
- Linear system of equations** $A\vec{x} = \vec{b}$, m equations, n unknowns.
- Consistent system** $A\vec{x} = \vec{b}$: for every \vec{b} there is exactly one solution \vec{x} .
- Vector form of linear equation** $x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{b}$, \vec{v}_i columns of A .
- Matrix form of linear equation** $\vec{w}_i \cdot \vec{x} = b_i$, \vec{w}_i rows of A .
- Augmented matrix** of $A\vec{x} = \vec{b}$ is the matrix $[A|\vec{b}]$ which has one column more as A .
- Coefficient matrix** of $A\vec{x} = \vec{b}$ is the matrix A .
- Partitioned matrix** matrix, where each entry is a matrix.
- Matrix multiplication** $[AB]_{ij} = \sum_k A_{ik}B_{kj}$, dot product of i -th row of A with j 'th column of B .
- Gauss-Jordan elimination** $A \rightarrow \text{rref}(A)$ in row reduced echelon form.
- Gauss-Jordan elimination steps**: swapping rows, scaling rows, adding rows to other rows.
- Row reduced echelon form**: every nonzero row has leading 1, columns with leading 1 have only 0 entries beside leading 1, every pivotal column to the left has leading 1 further up.
- Pivot column** column with leading 1 in $\text{rref}(A)$.
- Rank of matrix** A . Number of leading 1 in $\text{rref}(A)$. It is equal to $\dim(\text{im}(A))$.
- Nullety of matrix** A . Is defined as $\dim(\ker(A))$.
- Kernel of linear transformation** $\{\vec{x} \in \mathbf{R}^n, A\vec{x} = \vec{0}\}$.
- Image of linear transformation** $\{A\vec{x}, \vec{x} \in \mathbf{R}^n\}$.
- Inverse Linear transformation** satisfying $S(T(x)) = x = T(S(x))$. Corresponding matrix $B = A^{-1}$.
- Rotation in plane** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$, counterclockwise rotation by angle α .
- Dilation in plane** $\vec{x} \mapsto \lambda\vec{x}$, also called scaling.
- Rotation-Dilation** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Scale by $\sqrt{a^2 + b^2}$, rotate by $\arctan(b/a)$.
- Horizontal and vertical shear** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$.
- Shear** T leaves \vec{v} invariant and $A\vec{x} - \vec{x}$ parallel to \vec{v} . Shear-Check: all $A\vec{x} - \vec{x}$ are parallel.
- Reflection at line** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$.
- Projection onto line containing unit vector** $T(\vec{x}) = (\vec{x} \cdot \vec{v})\vec{v}$.
- $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **span** X : Every $\vec{x} \in X$ can be written as $\vec{x} = \sum_i a_i \vec{v}_i$.
- $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **linear independent** X : $\vec{0} = \sum_i a_i \vec{v}_i \Rightarrow a_1 = \dots = a_n = 0$.
- $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **basis** in X : linear independent in X and span X .
- dimension of linear space** X number of elements in a basis of X .
- \mathcal{B} **coordinates** $[\vec{v}]_{\mathcal{B}} = S^{-1}\vec{v}$, where $S = [\vec{v}_1, \dots, \vec{v}_n]$ contains basis vectors \vec{v}_i as columns.
- \mathcal{B} **matrix** of T The matrix is $B = S^{-1}AS$.

RESULTS.

- Linear transformations.** T is linear: $T(\vec{0}) = \vec{0}, T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}), T(\lambda\vec{x}) = \lambda T(\vec{x})$
- Solution.** A linear system of equations has either exactly 1, no or infinitely many solutions.
- Dimension formula.** $\dim(\ker(A)) + \dim(\text{im}(A)) = n$, where A is $m \times n$ matrix.
- Behavior of kernel under elimination** kernel stays invariant under Gauss-Jordan elimination.
- Behavior of image under elimination** image in general changes during Gauss-Jordan elimination.
- Basis of image** of A pivot columns of A form a basis of the image of A .
- Basis of kernel** of A introduce free variables for each non-Pivot column of A .
- Inverse** of 2×2 matrix switch diagonal elements, change sign of wings and divide by determinant.
- Kernel of composition** kernel of A is contained in the kernel of BA .
- Image of composition** image of BA is contained in the image of B .
- Matrix algebra** $(AB)^{-1} = B^{-1}A^{-1}, A(B + C) = AB + AC$, etc. Note: $AB \neq BA$ in general.
- A invertible** $\Leftrightarrow \text{rref}(A) = I \Leftrightarrow$ columns basis $\Leftrightarrow A\vec{x} = \vec{b}$ consistent $\Leftrightarrow \text{rank}(A) = n \Leftrightarrow \ker(A) = \{\vec{0}\}$.

PROBLEM You know $B = \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find the nullety and rank of A . Can you find the kernel of A ? Can you find the image of A .

SOLUTION. We can find the kernel but not the image. The dimensions of the image and kernel are 2 (2 pivot, 2 nonpivot columns). The linear system to B is $x + 2y + 5z = 0, z + 3u = 0$. Solving gives $u = t, z = -3t, y = s, x = -2s - 5t$, so that a general kernel element is $t[-5, 0, -3, 1] + s[-2, 1, 0, 0]$.

PROBLEM Given the basis $\vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ in the plane. a) Find the coordinates of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in this basis. b) Given a vector with coordinates $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in that basis, find the coordinates in the standard basis.

SOLUTION. $S = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$ has the inverse $S^{-1} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$. a) The vector has new coordinates $S^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ 7 \end{bmatrix}$, b) The vector has the standard coordinates $S \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$.

PROBLEM Let A be a shear along a line L . Find $\ker(A - I_2)$, $\text{im}(A - I_2)$ and $(A - I_2)^2$.

SOLUTION. Shears have the property that $(A - I)x = Ax - x$ is parallel to L , therefore $\text{im}(A - I)$ is L . The kernel of $A - I$ consists of vectors $Ax = x$. Because every $v \in L$ has this property, the kernel is L too. Both have dimension 1. We have $(A - I)^2 = 0$ because the image of $A - I$ is a subset of the kernel.

PROBLEM Let L be the line spanned by $\vec{v} = (1/2, 1/2, 0)$ and let T be the counterclockwise rotation about an angle $\pi/2$ around L (this means $\pi/2$ clockwise if you look from \vec{v} to the origin). Find the matrix A .

SOLUTION. Draw a good picture. \vec{e}_1 goes to $[1/2, 1/2, 1/\sqrt{2}]$, \vec{e}_2 goes to $[1/2, 1/2, -1/\sqrt{2}]$, \vec{e}_3 goes to $[-1/\sqrt{2}, 1/\sqrt{2}, 0]$. These are the columns of A , so that $A = \begin{bmatrix} 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$.

PROBLEM. Let A be a 3×3 matrix satisfying $A^2 = 0$. Show that the image of A is a subset of the kernel of A and determine all possible values for $\text{rank}(A)$. Give an example.

SOLUTION. If x is in the image then $x = Ay$ and $Ax = AAy = 0$ so that x is in the kernel. A is not invertible and can not have rank 3. It can be the 0 matrix with rank 0. It can have rank 1: take 0 in the first two columns and \vec{e}_1 in the last. It can not have rank 2 because the dimension of the kernel would be 1 and could not contain the kernel.

PROBLEM. Let $A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$ Find B satisfying $B^2 = A$, determine $\text{rank}(B)$ and B^{17} .

SOLUTION. A is a rotation-dilation, a composition of a rotation by $\pi/2$ and dilation by 4. Take B as a rotation dilation with angle $\pi/4$ and dilation factor 2. The rank of B is 2 because if it were smaller, then also the rank of A were smaller. B^{17} is a rotation dilation with angle $17\pi/2 \sim \pi/2$ and dilation factor 2^{17} .

PROBLEM. If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in \mathbf{R}^n are linearly independent, are $\vec{w}_1 = \vec{v}_1, \vec{w}_2 = \vec{v}_1 + \vec{v}_2, \vec{w}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ also linearly independent?

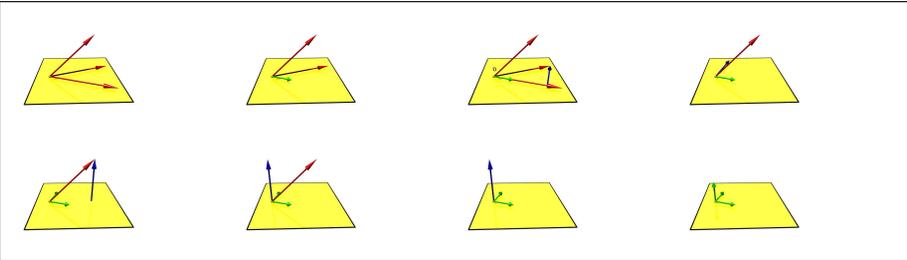
SOLUTION. Yes: the linear map which maps the span of \vec{v}_i into the span of \vec{w}_i is invertible: it has the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

PROBLEM. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & \dots & 100 \\ 101 & 102 & 103 & \dots & 200 \\ \dots & \dots & \dots & \dots & \dots \\ 901 & 902 & 903 & \dots & 1000 \end{bmatrix}$.

SOLUTION. Deleting the first row from each other row makes the lower 99 rows all linearly dependent. The rank is 2.

GRAM-SCHMIDT PROCESS.

Let $\vec{v}_1, \dots, \vec{v}_n$ be a basis in V . Let $\vec{u}_1 = \vec{v}_1$ and $\vec{w}_1 = \vec{u}_1 / \|\vec{u}_1\|$. The Gram-Schmidt process recursively constructs from the already constructed orthonormal set $\vec{w}_1, \dots, \vec{w}_{i-1}$ which spans a linear space V_{i-1} the new vector $\vec{u}_i = (\vec{v}_i - \text{proj}_{V_{i-1}}(\vec{v}_i))$ which is orthogonal to V_{i-1} , and then normalizing \vec{u}_i to get $\vec{w}_i = \vec{u}_i / \|\vec{u}_i\|$. Each vector \vec{w}_i is orthonormal to the linear space V_{i-1} . The vectors $\{\vec{w}_1, \dots, \vec{w}_n\}$ form an orthonormal basis in V .



EXAMPLE.

Find an orthonormal basis for $\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

SOLUTION.

- $\vec{w}_1 = \vec{v}_1 / \|\vec{v}_1\| = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
- $\vec{u}_2 = (\vec{v}_2 - \text{proj}_{V_1}(\vec{v}_2)) = \vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1 = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$. $\vec{w}_2 = \vec{u}_2 / \|\vec{u}_2\| = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
- $\vec{u}_3 = (\vec{v}_3 - \text{proj}_{V_2}(\vec{v}_3)) = \vec{v}_3 - (\vec{w}_1 \cdot \vec{v}_3)\vec{w}_1 - (\vec{w}_2 \cdot \vec{v}_3)\vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$, $\vec{w}_3 = \vec{u}_3 / \|\vec{u}_3\| = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

QR FACTORIZATION.

The formulas can be written as

$$\vec{v}_1 = \|\vec{v}_1\| \vec{w}_1 = r_{11} \vec{w}_1$$

...

$$\vec{v}_i = (\vec{w}_1 \cdot \vec{v}_i)\vec{w}_1 + \dots + (\vec{w}_{i-1} \cdot \vec{v}_i)\vec{w}_{i-1} + \|\vec{u}_i\| \vec{w}_i = r_{i1}\vec{w}_1 + \dots + r_{ii}\vec{w}_i$$

...

$$\vec{v}_n = (\vec{w}_1 \cdot \vec{v}_n)\vec{w}_1 + \dots + (\vec{w}_{n-1} \cdot \vec{v}_n)\vec{w}_{n-1} + \|\vec{u}_n\| \vec{w}_n = r_{n1}\vec{w}_1 + \dots + r_{nn}\vec{w}_n$$

which means in matrix form

$$A = \begin{bmatrix} | & | & \cdot & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & | & \cdot & | \end{bmatrix} = \begin{bmatrix} | & | & \cdot & | \\ \vec{w}_1 & \dots & \vec{w}_m \\ | & | & \cdot & | \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1m} \\ 0 & r_{22} & \dots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_{mm} \end{bmatrix} = QR,$$

where A and Q are $n \times m$ matrices and R is a $m \times m$ matrix.

Any matrix A with linearly independent columns can be decomposed as $A = QR$, where Q has orthonormal column vectors and where R is an upper triangular square matrix.

BACK TO THE EXAMPLE. The matrix with the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$.

$\vec{w}_1 = \|\vec{v}_1\| \vec{w}_1$
 $\vec{v}_2 = (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1 + \|\vec{u}_2\| \vec{w}_2$
 $\vec{v}_3 = (\vec{w}_1 \cdot \vec{v}_3)\vec{w}_1 + (\vec{w}_2 \cdot \vec{v}_3)\vec{w}_2 + \|\vec{u}_3\| \vec{w}_3,$

so that $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$.

PRO MEMORIA.

While building the matrix R we keep track of the vectors u_i during the Gram-Schmidt procedure. At the end you have vectors $\vec{u}_i, \vec{v}_i, \vec{w}_i$ and the matrix R has $\|\vec{u}_i\|$ in the diagonal as well as the dot products $\vec{w}_i \cdot \vec{v}_j$ in the upper right triangle.

PROBLEM. Make the QR decomposition of $A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. $\vec{w}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. $\vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

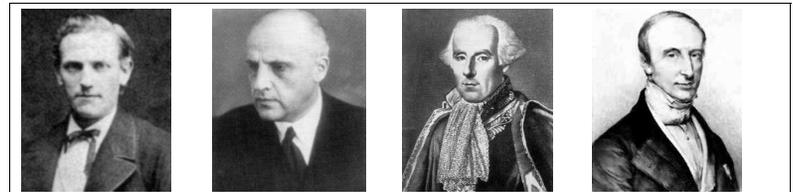
$\vec{w}_2 = \vec{u}_2$. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = QR$.

WHY do we care to have an orthonormal basis?

- An orthonormal basis looks like the standard basis $\vec{v}_1 = (1, 0, \dots, 0), \dots, \vec{v}_n = (0, 0, \dots, 1)$. Actually, we will see that an orthonormal basis into a standard basis or a mirror of the standard basis.
- The Gram-Schmidt process is tied to the factorization $A = QR$. The later helps to solve linear equations. In physical problems like in astrophysics, the numerical methods to simulate the problems one needs to invert huge matrices in every time step of the evolution. The reason why this is necessary sometimes is to assure the numerical method is stable implicit methods. Inverting $A^{-1} = R^{-1}Q^{-1}$ is easy because R and Q are easy to invert.
- For many physical problems like in quantum mechanics or dynamical systems, matrices are **symmetric** $A^* = A$, where $A_{ij}^* = \bar{A}_{ji}$. For such matrices, there will a natural orthonormal basis.
- The **formula for the projection** onto a linear subspace V simplifies with an orthonormal basis \vec{v}_j in V : $\text{proj}_V(\vec{x}) = (\vec{v}_1 \cdot \vec{x})\vec{w}_1 + \dots + (\vec{w}_n \cdot \vec{x})\vec{w}_n$.
- An orthonormal basis simplifies computations due to the presence of many zeros $\vec{w}_j \cdot \vec{w}_i = 0$. This is especially the case for problems with symmetry.
- The Gram Schmidt process can be used to define and construct classes of classical polynomials, which are important in physics. Examples are Chebyshev polynomials, Laguerre polynomials or Hermite polynomials.
- QR factorization allows fast computation of the determinant, least square solutions $R^{-1}Q^{-1}\vec{b}$ of overdetermined systems $A\vec{x} = \vec{b}$ or finding eigenvalues - all topics which will appear later.

SOME HISTORY.

The recursive formulae of the process were stated by Erhard Schmidt (1876-1959) in 1907. The essence of the formulae were already in a 1883 paper of J.P.Gram in 1883 which Schmidt mentions in a footnote. The process seems already have been used by Laplace (1749-1827) and was also used by Cauchy (1789-1857) in 1836.



Gram

Schmidt

Laplace

Cauchy

ORTHOGONAL MATRICES 10/27/2002

Math 21b, O. Knill

TRANSPOSE The **transpose** of a matrix A is the matrix $(A^T)_{ij} = A_{ji}$. If A is a $n \times m$ matrix, then A^T is a $m \times n$ matrix. For square matrices, the transposed matrix is obtained by reflecting the matrix at the diagonal.

EXAMPLES The transpose of a vector $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is the row vector $A^T = [1 \ 2 \ 3]$.

The transpose of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

A PROPERTY OF THE TRANSPOSE.

a) $\vec{x} \cdot \vec{A}y = A^T \vec{x} \cdot \vec{y}$.

b) $(AB)^T = B^T A^T$.

c) $(A^T)^T = A$.

PROOFS.

a) Because $x \cdot Ay = \sum_j \sum_i x_i A_{ij} y_j$ and $A^T x \cdot y = \sum_j \sum_i A_{ji} x_j y_i$ the two expressions are the same by renaming i and j .

b) $(AB)_{kl} = \sum_i A_{ki} B_{il}$. $(AB)_{kl}^T = \sum_i A_{il} B_{ki} = A^T B^T$.

c) $((A^T)^T)_{ij} = (A^T)_{ji} = A_{ij}$.

ORTHOGONAL MATRIX. A $n \times n$ matrix A is called **orthogonal** if $A^T A = 1$. The corresponding linear transformation is called **orthogonal**.

INVERSE. It is easy to invert an orthogonal matrix: $A^{-1} = A^T$.

EXAMPLES. The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ is orthogonal because its column vectors have length 1 and are orthogonal to each other. Indeed: $A^T A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. A reflection at a line is an orthogonal transformation because the columns of the matrix A have length 1 and are orthogonal. Indeed: $A^T A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \cdot \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

FACTS. An orthogonal transformation preserves the dot product: $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$. Proof: this is a homework assignment: Hint: just look at the properties of the transpose.

Orthogonal transformations preserve the **length** of vectors as well as the **angles** between them.

Proof. We have $\|A\vec{x}\|^2 = A\vec{x} \cdot A\vec{x} = \vec{x} \cdot \vec{x} = \|\vec{x}\|^2$. Let α be the angle between \vec{x} and \vec{y} and let β denote the angle between $A\vec{x}$ and $A\vec{y}$ and α the angle between \vec{x} and \vec{y} . Using $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ we get $\|A\vec{x}\| \|A\vec{y}\| \cos(\beta) = A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos(\alpha)$. Because $\|A\vec{x}\| = \|\vec{x}\|$, $\|A\vec{y}\| = \|\vec{y}\|$, this means $\cos(\alpha) = \cos(\beta)$. Because this property holds for all vectors we can rotate \vec{x} in plane V spanned by \vec{x} and \vec{y} by an angle ϕ to get $\cos(\alpha + \phi) = \cos(\beta + \phi)$ for all ϕ . Differentiation with respect to ϕ at $\phi = 0$ shows also $\sin(\alpha) = \sin(\beta)$ so that $\alpha = \beta$.

ORTHOGONAL MATRICES AND BASIS. A linear transformation A is orthogonal if and only if the column vectors of A form an orthonormal basis. (That is what $A^T A = 1_n$ means.)

COMPOSITION OF ORTHOGONAL TRANSFORMATIONS. The composition of two orthogonal transformations is orthogonal. The inverse of an orthogonal transformation is orthogonal. Proof. The properties of the transpose give $(AB)^T AB = B^T A^T AB = B^T B = 1$ and $(A^{-1})^T A^{-1} = (A^T)^{-1} A^{-1} = (AA^T)^{-1} = 1$.

EXAMPLES.

The composition of two reflections at a line is a rotation.

The composition of two rotations is a rotation.

The composition of a reflections at a plane with a reflection at an other plane is a rotation (the axis of rotation is the intersection of the planes).

ORTHOGONAL PROJECTIONS. The orthogonal projection P onto a linear space with orthonormal basis $\vec{v}_1, \dots, \vec{v}_n$ is the matrix $\begin{bmatrix} AA^T \end{bmatrix}$, where A is the matrix with column vectors \vec{v}_i . To see this just translate the formula $P\vec{x} = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + \dots + (\vec{v}_n \cdot \vec{x})\vec{v}_n$ into the language of matrices: $A^T \vec{x}$ is a vector with components $\vec{b}_i = (\vec{v}_i \cdot \vec{x})$ and $A\vec{b}$ is the sum of the $\vec{b}_i \vec{v}_i$, where \vec{v}_i are the column vectors of A . Orthogonal projections are no orthogonal transformations in general!

EXAMPLE. Find the orthogonal projection P from \mathbf{R}^3 to the linear space spanned by $\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} \frac{1}{5}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Solution: $AA^T = \begin{bmatrix} 0 & 1 \\ 3/5 & 0 \\ 4/5 & 0 \end{bmatrix} \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9/25 & 12/25 \\ 0 & 12/25 & 16/25 \end{bmatrix}$.

WHY ARE ORTHOGONAL TRANSFORMATIONS USEFUL?

- In Physics, Galileo transformations are compositions of translations with orthogonal transformations. The laws of classical mechanics are invariant under such transformations. This is a symmetry.
- Many coordinate transformations are orthogonal transformations. We will see examples when dealing with differential equations.
- In the QR decomposition of a matrix A , the matrix Q is orthogonal. Because $Q^{-1} = Q^t$, this allows to invert A easier.
- Fourier transformations are orthogonal transformations. We will see this transformation later in the course. In application, it is useful in computer graphics (i.e. JPG) and sound compression (i.e. MP3).
- Quantum mechanical evolutions (when written as real matrices) are orthogonal transformations.

WHICH OF THE FOLLOWING MAPS ARE ORTHOGONAL TRANSFORMATIONS?:

Yes	No	Shear in the plane.
Yes	No	Projection in three dimensions onto a plane.
Yes	No	Reflection in two dimensions at the origin.
Yes	No	Reflection in three dimensions at a plane.
Yes	No	Dilation with factor 2.
Yes	No	The Lorenz boost $\vec{x} \mapsto A\vec{x}$ in the plane with $A = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix}$
Yes	No	A translation.

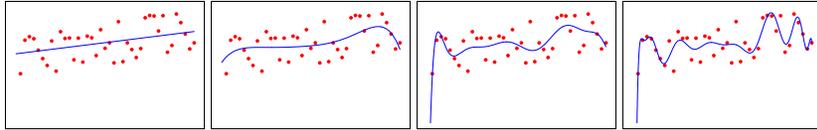
CHANGING COORDINATES ON THE EARTH. Problem: what is the matrix which rotates a point on earth with (latitude,longitude)=(a_1, b_1) to a point with (latitude,longitude)=(a_2, b_2)? Solution: The matrix which rotate the point $(0,0)$ to (a, b) a composition of two rotations. The first rotation brings the point into the right latitude, the second brings the point into the right longitude. $R_{a,b} = \begin{bmatrix} \cos(b) & -\sin(b) & 0 \\ \sin(b) & \cos(b) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(a) & 0 & -\sin(a) \\ 0 & 1 & 0 \\ \sin(a) & 0 & \cos(a) \end{bmatrix}$. To bring a point (a_1, b_1) to a point (a_2, b_2) , we form $A = R_{a_2, b_2} R_{a_1, b_1}^{-1}$.



EXAMPLE: With Cambridge (USA): $(a_1, b_1) = (42.366944, 288.893889)\pi/180$ and Zürich (Switzerland): $(a_2, b_2) = (47.377778, 8.551111)\pi/180$, we get the matrix

$$A = \begin{bmatrix} 0.178313 & -0.980176 & -0.0863732 \\ 0.983567 & 0.180074 & -0.0129873 \\ 0.028284 & -0.082638 & 0.996178 \end{bmatrix}$$

GOAL. The best possible "solution" of an inconsistent linear systems $Ax = b$ will be called the **least square solution**. It is the orthogonal projection of b onto the image $\text{im}(A)$ of A . The theory of the kernel and the image of linear transformations helps to understand this situation and leads to an explicit formula for the least square fit. Why do we care about non-consistent systems? Often we have to solve linear systems of equations with more constraints than variables. An example is when we try to find the best polynomial which passes through a set of points. This problem is called **data fitting**. If we wanted to accommodate all data, the degree of the polynomial would become too large, the fit would look too wiggly. Taking a smaller degree polynomial will not only be more convenient but also give a better picture. Especially important is **regression**, the fitting of data with lines.



The above pictures show 30 data points which are fitted best with polynomials of degree 1, 6, 11 and 16. The first linear fit maybe tells most about the trend of the data.

THE ORTHOGONAL COMPLEMENT OF $\text{im}(A)$. Because a vector is in the kernel of A^T if and only if it is orthogonal to the rows of A^T and so to the columns of A , the kernel of A^T is the orthogonal complement of $\text{im}(A)$: $(\text{im}(A))^\perp = \ker(A^T)$

EXAMPLES.

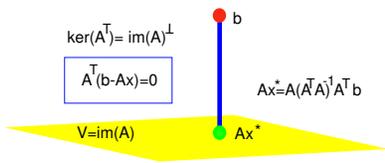
- $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. The kernel V of $A^T = [a \ b \ c]$ consists of all vectors satisfying $ax + by + cz = 0$. V is a plane. The orthogonal complement is the image of A which is spanned by the normal vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to the plane.
- $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. The image of A is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ the kernel of A^T is spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

ORTHOGONAL PROJECTION. If \vec{b} is a vector and V is a linear subspace, then $\text{proj}_V(\vec{b})$ is the vector closest to \vec{b} on V : given any other vector \vec{v} on V , one can form the triangle $\vec{b}, \vec{v}, \text{proj}_V(\vec{b})$ which has a right angle at $\text{proj}_V(\vec{b})$ and invoke Pythagoras.

THE KERNEL OF $A^T A$. For any $m \times n$ matrix $\ker(A) = \ker(A^T A)$ Proof. \subset is clear. On the other hand $A^T Av = 0$ means that Av is in the kernel of A^T . But since the image of A is orthogonal to the kernel of A^T , we have $A\vec{v} = 0$, which means \vec{v} is in the kernel of A .

LEAST SQUARE SOLUTION. The **least square solution** of $A\vec{x} = \vec{b}$ is the vector \vec{x}^* such that $A\vec{x}^*$ is closest to \vec{b} from all other vectors $A\vec{x}$. In other words, $A\vec{x}^* = \text{proj}_V(\vec{b})$, where $V = \text{im}(A)$. Because $\vec{b} - A\vec{x}^*$ is in $V^\perp = \text{im}(A)^\perp = \ker(A^T)$, we have $A^T(\vec{b} - A\vec{x}^*) = 0$. The last equation means that \vec{x}^* is a solution of $A^T A\vec{x} = A^T \vec{b}$, the **normal equation** of $A\vec{x} = \vec{b}$.

If the kernel of A is trivial, then the kernel of $A^T A$ is trivial and $A^T A$ can be inverted. Therefore $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$ is the least square solution.



WHY LEAST SQUARES? If \vec{x}^* is the least square solution of $A\vec{x} = \vec{b}$ then $\|A\vec{x}^* - \vec{b}\| \leq \|A\vec{x} - \vec{b}\|$ for all \vec{x} . Proof. $A^T(A\vec{x}^* - \vec{b}) = 0$ means that $A\vec{x}^* - \vec{b}$ is in the kernel of A^T which is orthogonal to $V = \text{im}(A)$. That is $\text{proj}_V(\vec{b}) = A\vec{x}^*$ which is the closest point to \vec{b} on V .

ORTHOGONAL PROJECTION If $\vec{v}_1, \dots, \vec{v}_n$ is a basis in V which is not necessarily orthonormal, then the orthogonal projection is $\vec{x} \mapsto A(A^T A)^{-1} A^T(\vec{x})$ where $A = [\vec{v}_1, \dots, \vec{v}_n]$.

Proof. $\vec{x} = (A^T A)^{-1} A^T \vec{b}$ is the least square solution of $A\vec{x} = \vec{b}$. Therefore $A\vec{x} = A(A^T A)^{-1} A^T \vec{b}$ is the vector in $\text{im}(A)$ closest to \vec{b} .

Special case: If $\vec{w}_1, \dots, \vec{w}_n$ is an orthonormal basis in V , we had seen earlier that AA^T with $A = [\vec{w}_1, \dots, \vec{w}_n]$ is the orthogonal projection onto V (this was just rewriting $A\vec{x} = (\vec{w}_1 \cdot \vec{x})\vec{w}_1 + \dots + (\vec{w}_n \cdot \vec{x})\vec{w}_n$ in matrix form.) This follows from the above formula because $A^T A = I$ in that case.

EXAMPLE Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$. The orthogonal projection onto $V = \text{im}(A)$ is $\vec{b} \mapsto A(A^T A)^{-1} A^T \vec{b}$. We have

$$A^T A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \text{ and } A(A^T A)^{-1} A^T = \begin{bmatrix} 1/5 & 2/5 & 0 \\ 2/5 & 4/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For example, the projection of $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is $\vec{x}^* = \begin{bmatrix} 2/5 \\ 4/5 \\ 0 \end{bmatrix}$ and the distance to \vec{b} is $1/\sqrt{5}$. The point \vec{x}^* is the point on V which is closest to \vec{b} .

Remember the formula for the distance of \vec{b} to a plane V with normal vector \vec{n} ? It was $d = |\vec{n} \cdot \vec{b}| / \|\vec{n}\|$. In our case, we can take $\vec{n} = [-2, 1, 0]$ this formula gives the distance $1/\sqrt{5}$. Let's check: the distance of \vec{x}^* and \vec{b} is $\| (2/5, -1/5, 0) \| = 1/\sqrt{5}$.

EXAMPLE. Let $A = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$. Problem: find the matrix of the orthogonal projection onto the image of A .

The image of A is a one-dimensional line spanned by the vector $\vec{v} = (1, 2, 0, 1)$. We calculate $A^T A = 6$. Then

$$A(A^T A)^{-1} A^T = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix} / 6 = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} / 6$$

DATA FIT. Find a quadratic polynomial $p(t) = at^2 + bt + c$ which best fits the four data points $(-1, 8), (0, 8), (1, 4), (2, 16)$.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 8 \\ 8 \\ 4 \\ 16 \end{bmatrix}, A^T A = \begin{bmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix} \text{ and } \vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

Software packages like Mathematica have already built in the facility to fit numerical data:

```

DataPoints = {{(-1,8)}, {(0,8)}, {(1,4)}, {(2,16)}}
f = Function[{y, Fit[DataPoints, {1, x, x^2}, x] / # -> y}]
Show[ {ListPlot[DataPoints], Plot[f[t], {t, -1, 2}]}];
Series[f[x], {x, 0, 2}]
    
```

The series expansion of f showed that indeed, $f(t) = 5 - t + 3t^2$ is indeed best quadratic fit. Actually, Mathematica does the same to find the fit then what we do: **"Solving" an inconsistent system of linear equations as best as possible.**

PROBLEM: Prove $\text{im}(A) = \text{im}(AA^T)$.

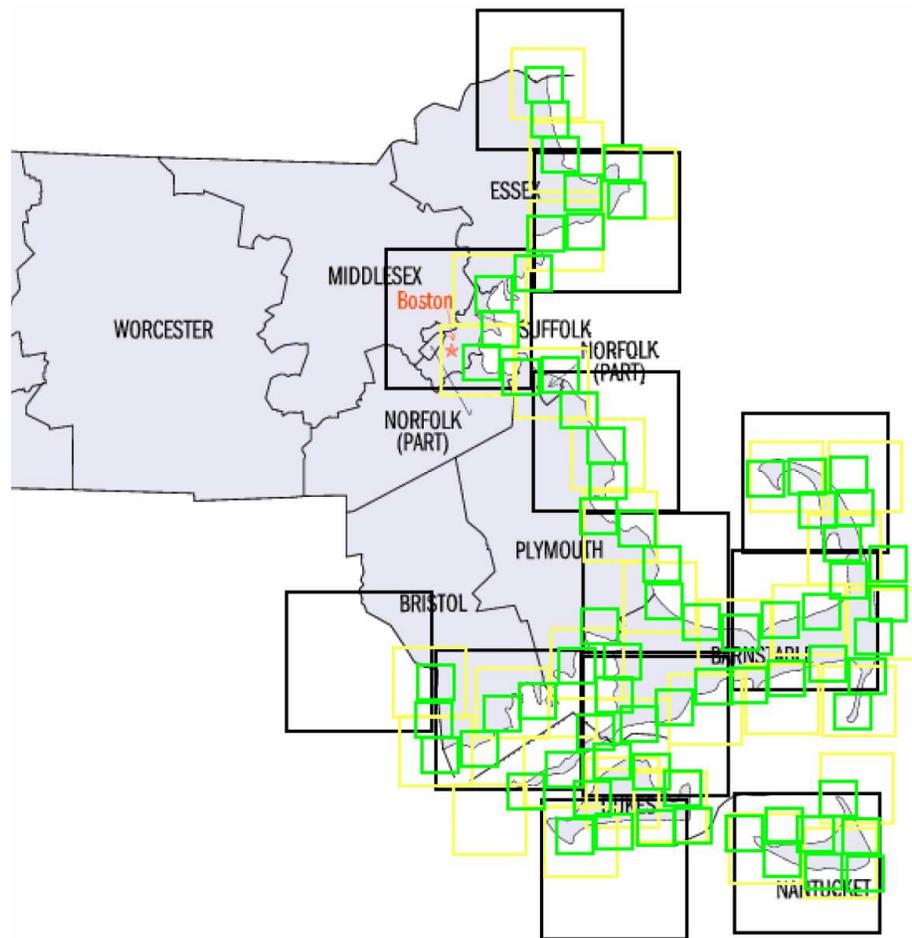
SOLUTION. The image of AA^T is contained in the image of A because we can write $\vec{v} = AA^T \vec{x}$ as $\vec{v} = A\vec{y}$ with $\vec{y} = A^T \vec{x}$. On the other hand, if \vec{v} is in the image of A , then $\vec{v} = A\vec{x}$. If $\vec{x} = \vec{y} + \vec{z}$, where \vec{y} is in the kernel of A and \vec{z} orthogonal to the kernel of A , then $A\vec{x} = A\vec{z}$. Because \vec{z} is orthogonal to the kernel of A , it is in the image of A^T . Therefore, $\vec{z} = A^T \vec{u}$ and $\vec{v} = A\vec{z} = AA^T \vec{u}$ is in the image of AA^T .

THE DIMENSION OF THE MASS COAST LINE

Math 21b, Fall 2003 O.Knill

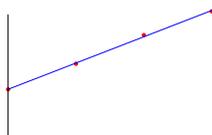
To measure the dimension of an object, one can count the number $f(n)$ of boxes of length $1/n$ needed to cover the object and see how $f(n)$ grows. If $f(n)$ grows like n^2 , then the dimension is 2, if $n(k)$ grows like n , the dimension is 1. For fractal objects, like coast lines, the number of boxes grows like n^s for a number s between 1 and 2. The dimension is obtained by correlating $y_k = \log_2 f(k)$ with $x_k = \log_2(k)$. We measure:

The Massachusetts coast line is a fractal of dimension 1.3.



We measure the data: $f(1) = 5$, $f(2) = 12$, $f(4) = 32$ and $f(8) = 72$. A plot of the data $(x_k, y_k) = (\log_2(k), \log_2 f(k))$ together with a **least square fit** can be seen to the right. The slope of the line is the dimension of the coast. It is about 1.295. In order to measure the dimension better, one would need better maps.

x_k	y_k
0	$\log_2(5)$
1	$\log_2(12)$
2	$\log_2(32)$
3	$\log_2(72)$



Finding the best linear fit $y = ax + b$ is equivalent to find the least square solution of the system

$$0a + b = \log_2(5), 1a + b = \log_2(12), 2a + b = \log_2(32), 3a + b = \log_2(72)$$

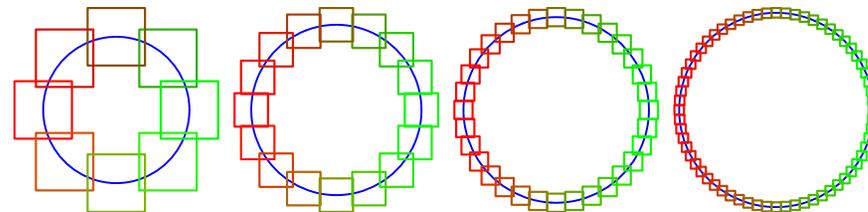
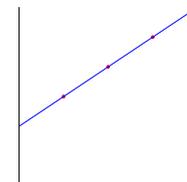
which is $A\vec{x} = \vec{b}$ with $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2.3 \\ 3.5 \\ 5 \\ 6 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$. We have $A^T A = \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix}$, $(A^T A)^{-1} =$

$$\begin{bmatrix} 2 & -3 \\ -3 & 7 \end{bmatrix} / 10 \text{ and } B = (A^T A)^{-1} A^T = \begin{bmatrix} -3 & -1 & 1 & 3 \\ 7 & 4 & 1 & -2 \end{bmatrix} / 10. \text{ We get } \begin{bmatrix} a \\ b \end{bmatrix} = B\vec{b} = \begin{bmatrix} 1.29 \\ 2.32 \end{bmatrix}.$$

COMPARISON: THE DIMENSION OF THE CIRCLE

Let us compare this with a smooth curve. To cover a circle with squares of size $2\pi/2^n$, we need about $f(n) = 2^n$ to cover the circle. We measure that the circle is not a fractal. It has dimension 1.

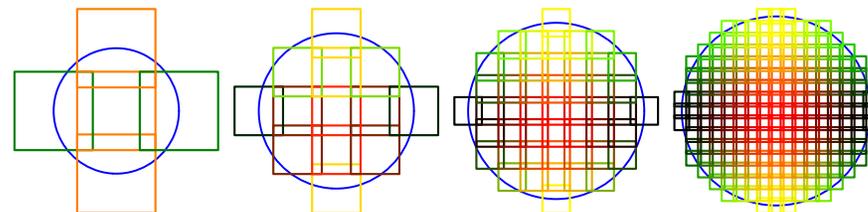
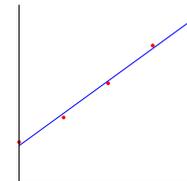
x_k	y_k
3	$\log_2(8) = 3$
4	$\log_2(16) = 4$
5	$\log_2(32) = 5$
6	$\log_2(64) = 6$



COMPARISON: THE DIMENSION OF THE DISK

For an other comparison, we take a disk. To cover the disk with squares of size $2\pi/2^n$, we need about $f(n) = 2^{2n}$ squares. We measure that the disk is not a fractal. It has dimension 2.

x_k	y_k
3	$\log_2(5) = 2.3$
4	$\log_2(13) = 3.7$
5	$\log_2(49) = 5.61$
6	$\log_2(213) = 7.73$



REMARKS

Calculating the dimension of coast lines is a classical illustration of "fractal theory". The coast of Brittain has been measured to have a dimension of 1.3 too. For natural objects, it is typical that measuring the length on a smaller scale gives larger results: one can measure empirically the dimension of mountains, clouds, plants, snowflakes, the lung etc. The dimension is an indicator how **rough** a curve or surface is.

DETERMINANTS I 10/31/2003

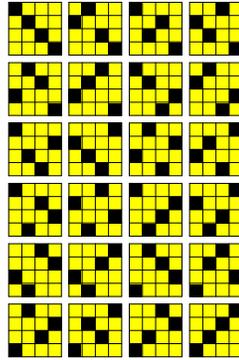
Math 21b, O. Knill

PERMUTATIONS. A **permutation** of n elements $\{1, 2, \dots, n\}$ is a rearrangement of $\{1, 2, \dots, n\}$. There are $n! = n \cdot (n-1) \cdot \dots \cdot 1$ different permutations of $\{1, 2, \dots, n\}$: fixing the position of first element leaves $(n-1)!$ possibilities to permute the rest.

EXAMPLE. There are 6 permutations of $\{1, 2, 3\}$: $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)$.

PATTERNS AND SIGN. The matrix A with zeros everywhere except $A_{i,\pi(i)} = 1$ is called a permutation matrix or the **pattern** of π . An **inversion** is a pair $k < l$ such that $\sigma(k) > \sigma(l)$. The **sign** of a permutation π , denoted by $(-1)^{\sigma(\pi)}$ is (-1) if there are an odd number of inversions in the pattern. Otherwise, the sign is defined to be $+1$.

EXAMPLES. $\sigma(1, 2) = 0, (-1)^\sigma = 1, \sigma(2, 1) = 1, (-1)^\sigma = -1$. The permutations $(1, 2, 3), (3, 2, 1), (2, 3, 1)$ have sign 1, the permutations $(1, 3, 2), (3, 2, 1), (2, 1, 3)$ have sign -1 .



DETERMINANT The **determinant** of a $n \times n$ matrix A is the sum $\sum_{\pi} (-1)^{\sigma(\pi)} A_{1\pi(1)} A_{2\pi(2)} \dots A_{n\pi(n)}$, where π runs over all permutations of $\{1, 2, \dots, n\}$ and $\sigma(\pi)$ is the sign of the permutation.

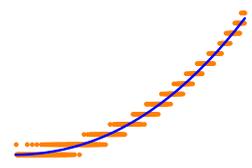
2 x 2 CASE. The determinant of $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is $ad - bc$. There are two permutations of $(1, 2)$. The identity permutation $(1, 2)$ gives $A_{11}A_{22}$, the permutation $(2, 1)$ gives $A_{21}A_{12}$. If you have seen some multi-variable calculus, you know that $\det(A)$ is the area of the parallelogram spanned by the column vectors of A . The two vectors form a basis if and only if $\det(A) \neq 0$.

3 x 3 CASE. The determinant of $A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is $aei + bfg + cdh - ceg - fha - bdi$ corresponding to the 6 permutations of $(1, 2, 3)$. Geometrically, $\det(A)$ is the volume of the parallelepiped spanned by the column vectors of A . The three vectors form a basis if and only if $\det(A) \neq 0$.

EXAMPLE DIAGONAL AND TRIANGULAR MATRICES. The determinant of a diagonal or triangular matrix is the product of the diagonal elements.

EXAMPLE PERMUTATION MATRICES. The determinant of a matrix which has everywhere zeros except $A_{i\pi(j)} = 1$ is just the sign $(-1)^{\sigma(\pi)}$ of the permutation.

HOW FAST CAN WE COMPUTE THE DETERMINANT?



The cost to find the determinant is the same as for the Gauss-Jordan elimination as we will see below. The graph to the left shows some measurements of the time needed for a CAS to calculate the determinant in dependence on the size of the $n \times n$ matrix. The matrix size ranges from $n=1$ to $n=300$. We also see a best cubic fit of these data using the least square method from the last lesson. It is the cubic $p(x) = a + bx + cx^2 + dx^3$ which fits best through the 300 data points.

WHY DO WE CARE ABOUT DETERMINANTS?

- check invertibility of matrices
- have geometric interpretation as volume
- explicit algebraic expressions for inverting a matrix
- as a natural functional on matrices it appears in formulas in particle or statistical physics
- allow to define orientation in any dimensions
- appear in change of variable formulas in higher dimensional integration.
- proposed alternative concepts are unnatural, hard to teach and harder to understand
- determinants are fun

TRIANGULAR AND DIAGONAL MATRICES.

The determinant of a **diagonal** or **triangular** matrix is the product of its diagonal elements.

Example: $\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix} = 20$.

PARTITIONED MATRICES.

The determinant of a **partitioned matrix** $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is the product $\det(A)\det(B)$.

Example $\det \begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix} = 2 \cdot 12 = 24$.

LINEARITY OF THE DETERMINANT. If the columns of A and B are the same except for the i 'th column,

$$\det([v_1, \dots, v, \dots, v_n]) + \det([v_1, \dots, w, \dots, v_n]) = \det([v_1, \dots, v+w, \dots, v_n])$$

In general, one has

$$\det([v_1, \dots, kv, \dots, v_n]) = k \det([v_1, \dots, v, \dots, v_n])$$

The same holds for rows. These identities follow directly from the original definition of the determinant.

PROPERTIES OF DETERMINANTS.

$$\det(AB) = \det(A)\det(B) \quad \det(SAS^{-1}) = \det(A)$$

$$\det(A^{-1}) = \det(A)^{-1} \quad \det(A^T) = \det(A)$$

If B is obtained from A by switching two rows, then $\det(B) = -\det(A)$. If B is obtained by adding an other row to a given row, then this does not change the value of the determinant.

PROOF OF $\det(AB) = \det(A)\det(B)$, one brings the $n \times n$ matrix $[A|AB]$ into row reduced echelon form. Similar than the augmented matrix $[A|b]$ was brought into the form $[1|A^{-1}b]$, we end up with $[1|A^{-1}AB] = [1|B]$. By looking at the $n \times n$ matrix to the left during Gauss-Jordan elimination, the determinant has changed by a factor $\det(A)$. We end up with a matrix B which has determinant $\det(B)$. Therefore, $\det(AB) = \det(A)\det(B)$. **PROOF OF $\det(A^T) = \det(A)$.** The transpose of a pattern is a pattern with the same signature.

PROBLEM. Find the determinant of $A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \end{bmatrix}$.

SOLUTION. Three row transpositions give $B = \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 7 & 2 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ a matrix which has determinant 84. Therefore $\det(A) = (-1)^3 \det(B) = -84$.

PROBLEM. Determine $\det(A^{100})$, where A is the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 16 \end{bmatrix}$.

SOLUTION. $\det(A) = 10, \det(A^{100}) = (\det(A))^{100} = 10^{100} = 1 \cdot \text{gogool}$. This name as well as the gogoolplex = $10^{10^{100}}$ are official. They are huge numbers: the mass of the universe for example is $10^{52}kg$ and $1/10^{10^{51}}$ is the chance to find yourself on Mars by quantum fluctuations. (R.E. Crandall, Scient. Amer., Feb. 1997).

ROW REDUCED ECHELON FORM. Determining $\text{rref}(A)$ also determines $\det(A)$.

If A is a matrix and α_i are the factors which are used to scale different rows and s is the number of times, two rows are switched, then $\det(A) = (-1)^s \alpha_1 \dots \alpha_n \det(\text{rref}(A))$.

INVERTIBILITY. Because of the last formula: A $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

THE LAPLACE EXPANSION.

In order to calculate by hand the determinant of $n \times n$ matrices $A = a_{ij}$ for $n > 3$, the following expansion is useful. Choose a column i . For each entry a_{ji} in that column, take the $(n-1) \times (n-1)$ matrix A_{ij} called **minor** which does not contain the i 'th column and j 'th row. Then

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \dots + (-1)^{i+n} a_{in} \det(A_{in}) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij})$$

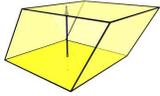
ORTHOGONAL MATRICES. Because $Q^T Q = 1$, we have $\det(Q)^2 = 1$ and so $|\det(Q)| = 1$. Rotations have determinant 1, reflections have determinant -1 .

QR DECOMPOSITION. If $A = QR$, then $\det(A) = \det(Q)\det(R)$. The determinant of Q is ± 1 , the determinant of R is the product of the diagonal elements of R .

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DETERMINANT AND VOLUME. If A is a $n \times n$ matrix, then $|\det(A)|$ is the volume of the n -dimensional parallelepiped E_n spanned by the n column vectors v_j of A .



Proof. Use the QR decomposition $A = QR$, where Q is orthogonal and R is upper triangular. From $QQ^T = I$, we get $1 = \det(Q)\det(Q^T) = \det(Q)^2$ see that $|\det(Q)| = 1$. Therefore, $\det(A) = \pm \det(R)$. The determinant of R is the product of the $\|u_i\| = \|v_i - \text{proj}_{V_{j-1}} v_i\|$ which was the distance from v_i to V_{j-1} . The volume $\text{vol}(E_j)$ of a j -dimensional parallelepiped E_j with base E_{j-1} in V_{j-1} and height $\|u_j\|$ is $\text{vol}(E_{j-1})\|u_j\|$. Inductively $\text{vol}(E_j) = \|u_j\|\text{vol}(E_{j-1})$ and therefore $\text{vol}(E_n) = \prod_{j=1}^n \|u_j\| = \det(R)$.

The volume of a k dimensional parallelepiped defined by the vectors v_1, \dots, v_k is $\sqrt{|\det(A^T A)|}$.

Proof. $Q^T Q = I_n$ gives $A^T A = (QR)^T(QR) = R^T Q^T Q R = R^T R$. So, $\det(R^T R) = \det(R)^2 = (\prod_{j=1}^k \|u_j\|)^2$. (Note that A is a $n \times k$ matrix and that $A^T A = R^T R$ and R are $k \times k$ matrices.)

ORIENTATION. Determinants allow to **define** the orientation of n vectors in n -dimensional space. This is "handy" because there is no "right hand rule" in hyperspace... To do so, define the matrix A with column vectors v_j and define the orientation as the sign of $\det(A)$. In three dimensions, this agrees with the right hand rule: if v_1 is the thumb, v_2 is the pointing finger and v_3 is the middle finger, then their orientation is positive.

$x_i \det(A) =$

CRAMER'S RULE. This is an explicit formula for the solution of $A\vec{x} = \vec{b}$. If A_i denotes the matrix, where the column \vec{v}_i of A is replaced by \vec{b} , then

$$x_i = \det(A_i) / \det(A)$$

Proof. $\det(A_i) = \det([v_1, \dots, b, \dots, v_n]) = \det([v_1, \dots, (Ax), \dots, v_n]) = \det([v_1, \dots, \sum_i x_i v_i, \dots, v_n]) = x_i \det([v_1, \dots, v_i, \dots, v_n]) = x_i \det(A)$

EXAMPLE. Solve the system $5x+3y = 8, 8x+5y = 2$ using Cramers rule. This linear system with $A = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$. We get $x = \det \begin{bmatrix} 8 & 3 \\ 2 & 5 \end{bmatrix} = 34y = \det \begin{bmatrix} 5 & 8 \\ 8 & 2 \end{bmatrix} = -54$.

GABRIEL CRAMER. (1704-1752), born in Geneva, Switzerland, he worked on geometry and analysis. Cramer used the rule named after him in a book "Introduction à l'analyse des lignes courbes algébrique", where he solved like this a system of equations with 5 unknowns. According to a short biography of Cramer by J.J O'Connor and E F Robertson, the rule had however been used already before by other mathematicians. Solving systems with Cramers formulas is slower than by Gaussian elimination. The rule is still important. For example, if A or b depends on a parameter t , and we want to see how x depends on the parameter t one can find explicit formulas for $(d/dt)x_i(t)$.

THE INVERSE OF A MATRIX. Because the columns of A^{-1} are solutions of $A\vec{x} = \vec{e}_i$, where \vec{e}_j are basis vectors, Cramers rule together with the Laplace expansion gives the formula:

$$[A^{-1}]_{ij} = (-1)^{i+j} \det(A_{ji}) / \det(A)$$

$B_{ij} = (-1)^{i+j} \det(A_{ji})$ is called the **classical adjoint** of A . **Note** the change $ij \rightarrow ji$. **Don't** confuse the classical adjoint with the **transpose** A^T which is sometimes also called the **adjoint**.

EXAMPLE. $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 4 \\ 6 & 0 & 7 \end{bmatrix}$ has $\det(A) = -17$ and we get $A^{-1} = \begin{bmatrix} 14 & -21 & 10 \\ -11 & 8 & -3 \\ -12 & 18 & -11 \end{bmatrix} / (-17)$:

$B_{11} = (-1)^{1+1} \det \begin{bmatrix} 2 & 4 \\ 0 & 7 \end{bmatrix} = 14$. $B_{12} = (-1)^{1+2} \det \begin{bmatrix} 3 & 1 \\ 0 & 7 \end{bmatrix} = -21$. $B_{13} = (-1)^{1+3} \det \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = 10$.

$B_{21} = (-1)^{2+1} \det \begin{bmatrix} 5 & 4 \\ 6 & 7 \end{bmatrix} = -11$. $B_{22} = (-1)^{2+2} \det \begin{bmatrix} 2 & 1 \\ 6 & 7 \end{bmatrix} = 8$. $B_{23} = (-1)^{2+3} \det \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} = -3$.

$B_{31} = (-1)^{3+1} \det \begin{bmatrix} 5 & 2 \\ 6 & 0 \end{bmatrix} = -12$. $B_{32} = (-1)^{3+2} \det \begin{bmatrix} 2 & 3 \\ 6 & 0 \end{bmatrix} = 18$. $B_{33} = (-1)^{3+3} \det \begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} = -11$.

THE ART OF CALCULATING DETERMINANTS. When confronted with a matrix, it is good to go through a checklist of methods to crack the determinant. Often, there are different possibilities to solve the problem, in many cases the solution is particularly simple using one method.

- Is it a 2×2 or 3×3 matrix?
- Do you see duplicated columns or rows?
- Is it an upper or lower triangular matrix?
- Can you row reduce to a triangular case?
- Is it a partitioned matrix?
- Are there only a few nonzero patters?
- Is it a trick: like A^{1000} ?
- Laplace expansion with some row or column?
- Does geometry imply noninvertibility?
- Later: Can you see the eigenvalues of $A - \lambda I_n$?

EXAMPLES.

1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 5 & 5 & 5 & 5 & 4 \\ 1 & 3 & 2 & 7 & 4 \\ 3 & 2 & 8 & 4 & 9 \end{bmatrix}$ Try row reduction.

2) $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$ Laplace expansion.

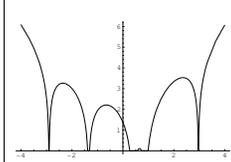
3) $A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$ Partitioned matrix.

4) $A = \begin{bmatrix} 1 & 6 & 10 & 1 & 15 \\ 2 & 8 & 17 & 1 & 29 \\ 0 & 0 & 3 & 8 & 12 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ Make it tridiagonal.

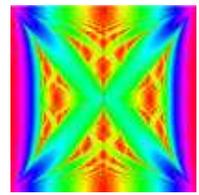
APPLICATION HOFSTADTER BUTTERFLY. In solid state physics, one is interested in the function $f(E) = \det(L - EI_n)$, where

$$L = \begin{bmatrix} \lambda \cos(\alpha) & 1 & 0 & \dots & 0 & \dots & 1 \\ 1 & \lambda \cos(2\alpha) & 1 & \dots & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 & \dots & 0 \\ 0 & \dots & \dots & 1 & \lambda \cos((n-1)\alpha) & \dots & 1 \\ 1 & 0 & \dots & 0 & 1 & \dots & \lambda \cos(n\alpha) \end{bmatrix}$$

describes an electron in a periodic crystal, E is the energy and $\alpha = 2\pi/n$. The electron can move as a Bloch wave whenever the determinant is negative. These intervals form the **spectrum** of the quantum mechanical system. A physicist is interested in the rate of change of $f(E)$ or its dependence on λ when E is fixed.



The graph to the left shows the function $E \mapsto \log(|\det(L - EI_n)|)$ in the case $\lambda = 2$ and $n = 5$. In the energy intervals, where this function is zero, the electron can move, otherwise the crystal is an insulator. The picture to the right shows the spectrum of the crystal depending on α . It is called the "Hofstadter butterfly" made popular in the book "Gödel, Escher Bach" by Douglas Hofstadter.



EIGENVALUES AND EIGENVECTORS. A nonzero vector v is called an **eigenvector** of A with **eigenvalue** λ if $[Av = \lambda v]$.

EXAMPLES.

- \vec{v} is an eigenvector to the eigenvalue 0 if \vec{v} is in the kernel of A .
- A shear A in the direction v has an eigenvector \vec{v} .
- Projections have eigenvalues 1 or 0.
- Reflections have eigenvalues 1 or -1 .
- If A is a diagonal matrix with diagonal elements a_i , \vec{e}_i is an eigenvector with eigenvalue a_i .
- A rotation in the plane by an angle 30 degrees has no eigenvector. (The actual eigenvectors are complex).

LINEAR DYNAMICAL SYSTEMS.

Iterating a linear map $x \mapsto Ax$ is called a **discrete dynamical system**. One wants to understand what happens with $x_1 = Ax, x_2 = AAx = A^2x, x_3 = AAAx = A^3x, \dots$

EXAMPLE 1: $x \mapsto ax$ or $x_{n+1} = ax_n$ has the solution $x_n = a^n x_0$. For example, $1.03^{20} \cdot 1000 = 1806.11$ is the balance on a bank account which had 1000 dollars 20 years ago and if the interest rate was constant 3 percent.

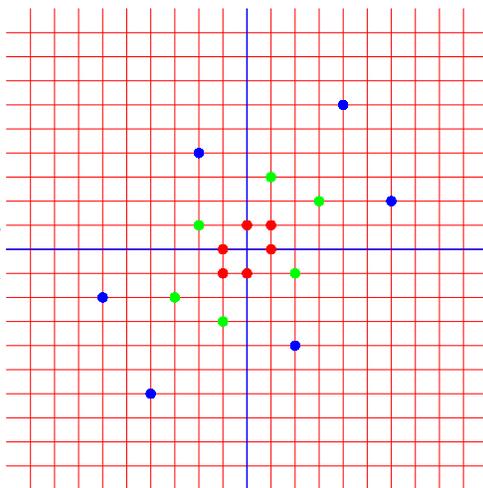
EXAMPLE 2: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $A\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A^2\vec{v} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. $A^3\vec{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. $A^4\vec{v} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ etc.

EXAMPLE 3: If \vec{v} is an eigenvector with eigenvalue λ , then $A\vec{v} = \lambda\vec{v}, A^2\vec{v} = A(A\vec{v}) = A\lambda\vec{v} = \lambda A\vec{v} = \lambda^2\vec{v}$ and more generally $A^n\vec{v} = \lambda^n\vec{v}$.

RECURSION: If a scalar quantity u_{n+1} does not only depend on u_n but also on u_{n-1} we can write $(x_n, y_n) = (u_n, u_{n-1})$ and get a linear map because x_{n+1}, y_{n+1} depend in a linear way on x_n, y_n .

A RECURSION PROBLEM. A linear recursion problem which appears in quantum mechanics is $u_{n+1} + u_{n-1} = Eu_n$ and $u_0 = 0, u_1 = 1$. Because $\begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix}$. The recursion is done by iterating the matrix A . Lets take $E = 1$:
 $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$
 $A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. We see that A^3 is a reflection at the origin which has the eigenvalue $\lambda = -1$ and A^6 is the identity. Every initial vector is mapped after 6 iterations back to its original starting point.

If the E parameter is changed, the dynamics also changes. For $E = 3$ for example, most initial points will escape to infinity similar as in the next example. Indeed, for $E = 3$, there is an eigenvector $\vec{v} = (3 + \sqrt{5})/2$ to the eigenvalue $\lambda = (3 + \sqrt{5})/2$ and $A^n\vec{v} = \lambda^n\vec{v}$ escapes to ∞ .

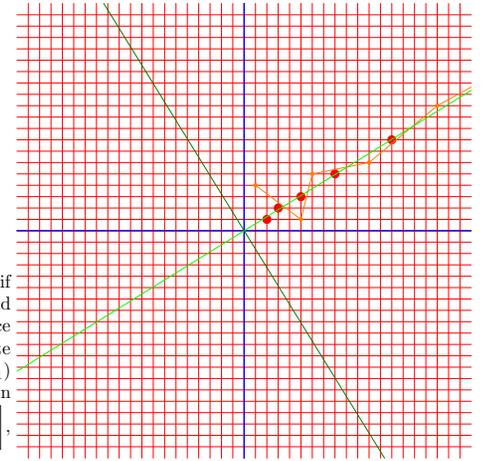


THE FIBONNACCI RECURSION: In the third section of Liber abacci, published in 1202 **Leonardo Fibonacci** (1170-1250) writes:



A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

Mathematically, how does u_n grow, if $u_{n+1} = u_n + u_{n-1}$? We can assume $u_0 = 1$ and $u_1 = 2$ to match Leonardos example. The sequence is $(1, 2, 3, 5, 8, 13, 21, \dots)$. As before we can write this recursion using vectors $(x_n, y_n) = (u_n, u_{n-1})$ starting with $(1, 2)$. The matrix A to this recursion is $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Iterating gives $A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = A \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.



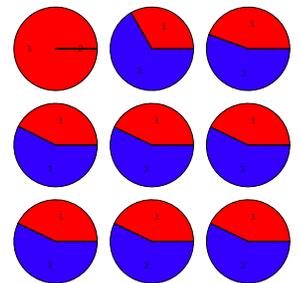
SOLUTION KNOWING EIGENSYSTEM. If $A\vec{v}_1 = \lambda_1\vec{v}_1, A\vec{v}_2 = \lambda_2\vec{v}_2$ and $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$, we have an explicit solution $A^n\vec{v} = c_1\lambda_1^n\vec{v}_1 + c_2\lambda_2^n\vec{v}_2$. This motivates to find good methods to compute eigenvalues and eigenvectors.

EVOLUTION OF QUANTITIES. Example could be market systems, population quantities of different species, or ingredient quantities in a chemical reaction. A linear description might not always be a good model but it has the advantage that we can solve the system explicitly. Eigenvectors will provide the key to do so. You do a biological problem like this in the homework.

EXAMPLE 1: **Quantum mechanics.** Some quantum mechanical systems of a particle in a potential V are described by $(Lu)_n = u_{n+1} + u_{n-1} + V_n u_n$. Energies E for which $(Lu)_n = Eu_n$, we have the recursion $u_{n+1} + u_{n-1} = (E - V_n)u_n$, when the potential is periodic in n , then this leads to a linear recursion problem. For example, if $V_n = V$ is constant, then $u_{n+1} + u_{n-1} = (E - V)u_n$. A question is for which E the solutions stay bounded. You have seen above the case $E - V = 1$.

EXAMPLE 2: **Chaos theory.** In plasma physics, one studies maps like $(x, y) \mapsto (2x - y - a \sin(x), x)$. You see that $(0, 0)$ is a fixed point. Near that fixed point, the map is described by its linearization $DT \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 2-a & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. For which a is this linear system stable near $(0, 0)$ in the sense that a point near $(0, 0)$ stays nearby for all times? The answer will be given using eigenvalues. Note that the matrix here is the same as in the quantum mechanical example before by putting $E = 2 - a$.

EXAMPLE 3: **Markov Processes.** The percentage of people using Apple OS or the Linux OS is represented by a vector $\begin{bmatrix} m \\ l \end{bmatrix}$. Each cycle 2/3 of Mac OS users switch to Linux and 1/3 stays. Also lets assume that 1/2 of the Linux OS users switch to apple and 1/2 stay. The matrix $P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$ encoding this dynamics is called a **Markov matrix**: the entries satisfy $0 \leq P_{ij} \leq 1$ and the sum of each column elements is equal to 1. What ratio of Apple/Linux users do we have after things settle to an equilibrium? We can simulate this with a dice: start in a state like $M = (1, 0)$ (all users have Macs). If the dice shows 3,4,5 or 6, a user in that group switch to Linux, otherwise stays in the M camp. Throw also a dice for each user in L. If 1,2 or 3 shows up, the user switches to M. The matrix P has an eigenvector $(3/7, 4/7)$ which belongs to the eigenvalue 1. The interpretation of $P\vec{v} = \vec{v}$ is that with this split up, there is no change in average.



RANDOMIZATION. Choose 5 random numbers between 1 and 6.

Step 1	Step 2	Step 3
A	B	C
-	D	E
-		

RULES.

If the vector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the upper number is smaller than 5 then switch to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ otherwise keep the vector.

If the vector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the lower number is even then switch to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ otherwise keep the vector.

RUN. We make three Markov steps. Depending on your random numbers, you will end up with a final vector.

\vec{v}	\vec{v}_1	\vec{v}_2	\vec{v}_3
	Step 1 use A	Step 2 use B or C	Step 3 use D or E
1			
0			

COMPUTING EIGENVALUES 11/7/2003

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THE TRACE. The **trace** of a matrix A is the sum of its diagonal elements.

EXAMPLES. The trace of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ is $1 + 4 + 8 = 13$. The trace of a skew symmetric matrix A is zero because there are zeros in the diagonal. The trace of I_n is n .

CHARACTERISTIC POLYNOMIAL. The polynomial $f_A(\lambda) = \det(\lambda I_n - A)$ is called the **characteristic polynomial** of A .

EXAMPLE. The characteristic polynomial of A above is $x^3 - 13x^2 + 15x$.

The eigenvalues of A are the roots of the characteristic polynomial $f_A(\lambda)$.

Proof. If λ is an eigenvalue of A with eigenfunction \vec{v} , then $A - \lambda I$ has \vec{v} in the kernel and $A - \lambda I$ is not invertible so that $f_A(\lambda) = \det(\lambda I - A) = 0$.

The polynomial has the form

$$f_A(\lambda) = \lambda^n - \text{tr}(A)\lambda^{n-1} + \dots + (-1)^n \det(A)$$

THE 2x2 CASE. The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$. The eigenvalues are $\lambda_{\pm} = T/2 \pm \sqrt{(T/2)^2 - D}$, where T is the trace and D is the determinant. In order that this is real, we must have $(T/2)^2 \geq D$. Away from that parabola, there are two different eigenvalues. The map A contracts volume for $|D| < 1$.

EXAMPLE. The characteristic polynomial of $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ is $\lambda^2 - 3\lambda + 2$ which has the roots 1, 2: $p_A(\lambda) = (\lambda - 1)(\lambda - 2)$.

THE FIBONNACCI RABBITS. The Fibonacci's recursion $u_{n+1} = u_n + u_{n-1}$ defines the growth of the rabbit population. We have seen that it can be rewritten as $\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = A \begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix}$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The roots of the characteristic polynomial $p_A(x) = \lambda^2 - \lambda - 1$ are $(\sqrt{5} + 1)/2, (\sqrt{5} - 1)/2$.

ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$ then λ is said to be an eigenvalue of **algebraic multiplicity** k .

EXAMPLE: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2 and the eigenvalue $\lambda = 2$ with algebraic multiplicity 1.

HOW TO COMPUTE EIGENVECTORS? Because $(\lambda - A)v = 0$, the vector v is in the kernel of $\lambda - A$. We know how to compute the kernel.

EXAMPLE FIBONNACCI. The kernel of $\lambda I_2 - A = \begin{bmatrix} \lambda_{\pm} - 1 & -1 \\ -1 & \lambda_{\pm} \end{bmatrix}$ is spanned by $\vec{v}_+ = [(1 + \sqrt{5})/2, 1]$ and $\vec{v}_- = [(1 - \sqrt{5})/2, 1]$. They form a basis B .

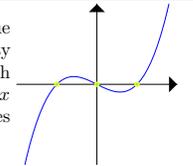
SOLUTION OF FIBONNACCI. To obtain a formula for $A^n \vec{v}$ with $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we form $[\vec{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{5}$.

Now, $\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = A^n \vec{v} = A^n (\vec{v}_+ / \sqrt{5} - \vec{v}_- / \sqrt{5}) = A^n \vec{v}_+ / \sqrt{5} - A^n \vec{v}_- / \sqrt{5} = \lambda_+^n \vec{v}_+ / \sqrt{5} + \lambda_-^n \vec{v}_- / \sqrt{5}$. We see that $u_n = [(\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n] / \sqrt{5}$.

ROOTS OF POLYNOMIALS.

For polynomials of degree 3 and 4 there exist explicit formulas in terms of radicals. As Galois (1811-1832) and Abel (1802-1829) have shown, it is not possible for equations of degree 5 or higher. Still, one can compute the roots numerically.

REAL SOLUTIONS. A $(2n + 1) \times (2n + 1)$ matrix A always has a real eigenvalue because the characteristic polynomial $p(x) = x^{2n+1} + \dots + \det(A)$ has the property that $p(x)$ goes to $\pm\infty$ for $x \rightarrow \pm\infty$. Because there exist values a, b for which $p(a) < 0$ and $p(b) > 0$, by the intermediate value theorem, there exists a real x with $p(x) = 0$. Application: A rotation in 11 dimensional space has all eigenvalues $|\lambda| = 1$. The real eigenvalue must have an eigenvalue 1 or -1 .



EIGENVALUES OF TRANSPOSE. We know (see homework) that the characteristic polynomials of A and A^T agree. Therefore A and A^T have the same eigenvalues.

APPLICATION: MARKOV MATRICES. A matrix A for which each column sums up to 1 is called a **Markov matrix**. The transpose of a Markov matrix has the eigenvector $\begin{bmatrix} 1 \\ 1 \\ \dots \\ 1 \end{bmatrix}$ with eigenvalue 1. Therefore:

A Markov matrix has an eigenvector \vec{v} to the eigenvalue 1.

This vector \vec{v} defines an equilibrium point of the Markov process.

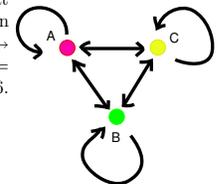
EXAMPLE. If $A = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix}$. Then $[3/7, 4/7]$ is the equilibrium eigenvector to the eigenvalue 1.

BRETSCHERS HOMETOWN. Problem 28 in the book deals with a Markov problem in Andelfingen the hometown of Bretscher, where people shop in two shops. (Andelfingen is a beautiful village at the Thur river in the middle of a "wine country"). Initially all shop in shop W . After a new shop opens, every week 20 percent switch to the other shop M . Missing something at the new place, every week, 10 percent switch back. This leads to a Markov matrix $A = \begin{bmatrix} 8/10 & 1/10 \\ 2/10 & 9/10 \end{bmatrix}$. After some time, things will settle down and we will have certain percentage shopping in W and other percentage shopping in M . This is the equilibrium.



MARKOV PROCESS IN PROBABILITY. Assume we have a graph like a network and at each node i , the probability to go from i to j in the next step is $[A]_{ij}$, where A_{ij} is a Markov matrix. We know from the above result that there is an eigenvector \vec{p} which satisfies $A\vec{p} = \vec{p}$. It can be normalized that $\sum_i p_i = 1$. The interpretation is that p_i is the probability that the walker is on the node p . For example, on a triangle, we can have the probabilities: $P(A \rightarrow B) = 1/2, P(A \rightarrow C) = 1/4, P(A \rightarrow A) = 1/4, P(B \rightarrow A) = 1/3, P(B \rightarrow B) = 1/6, P(B \rightarrow C) = 1/2, P(C \rightarrow A) = 1/2, P(C \rightarrow B) = 1/3, P(C \rightarrow C) = 1/6$. The corresponding matrix is

$$A = \begin{bmatrix} 1/4 & 1/3 & 1/2 \\ 1/2 & 1/6 & 1/3 \\ 1/4 & 1/2 & 1/6 \end{bmatrix}$$



In this case, the eigenvector to the eigenvalue 1 is $p = [38/107, 36/107, 33/107]^T$.

CALCULATING EIGENVECTORS 11/10/2003

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NOTATION. We often just write 1 instead of the identity matrix 1_n .

COMPUTING EIGENVALUES. Recall: because $\lambda - A$ has \vec{v} in the kernel if λ is an eigenvalue the characteristic polynomial $f_A(\lambda) = \det(\lambda - A) = 0$ has eigenvalues as roots.

2 x 2 CASE. Recall: The characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - (a+d)/2\lambda + (ad-bc)$. The eigenvalues are $\lambda_{\pm} = T/2 \pm \sqrt{(T/2)^2 - D}$, where $T = a + d$ is the trace and $D = ad - bc$ is the determinant of A . If $(T/2)^2 \geq D$, then the eigenvalues are real. Away from that parabola in the (T, D) space, there are two different eigenvalues. The map A contracts volume for $|D| < 1$.

NUMBER OF ROOTS. Recall: There are examples with no real eigenvalue (i.e. rotations). By inspecting the graphs of the polynomials, one can deduce that $n \times n$ matrices with odd n always have a real eigenvalue. Also $n \times n$ matrixes with even n and a negative determinant always have a real eigenvalue.

IF ALL ROOTS ARE REAL. $f_A(\lambda) = \lambda^n - \text{tr}(A)\lambda^{n-1} + \dots + (-1)^n \det(A) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$, we see that $\sum_i \lambda_i = \text{trace}(A)$ and $\prod_i \lambda_i = \det(A)$.

HOW TO COMPUTE EIGENVECTORS? Because $(\lambda - A)\vec{v} = 0$, the vector \vec{v} is in the kernel of $\lambda - A$.

EIGENVECTORS of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are \vec{v}_{\pm} with eigenvalue λ_{\pm} . If $c \neq 0$, then the eigenvectors to λ_{\pm} are $\vec{v}_{\pm} = \begin{bmatrix} \lambda_{\pm} - d \\ c \end{bmatrix}$.	If $c = 0$ and $d = 0$, then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors. If $c = 0$ and $a \neq d$, then the eigenvectors to a and d are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} b/(d-a) \\ 1 \end{bmatrix}$.
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ALGEBRAIC MULTIPLICITY. If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$, where $g(\lambda_0) \neq 0$, then f has **algebraic multiplicity** k . The algebraic multiplicity counts the number of times, an eigenvector occurs.

EXAMPLE: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2.

GEOMETRIC MULTIPLICITY. The dimension of the eigenspace E_{λ} of an eigenvalue λ is called the **geometric multiplicity** of λ .

EXAMPLE: the matrix of a shear is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It has the eigenvalue 1 with algebraic multiplicity 2. The kernel of $A - 1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the geometric multiplicity is 1. It is different from the algebraic multiplicity.

EXAMPLE: The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has eigenvalue 1 with algebraic multiplicity 2 and the eigenvalue 0 with multiplicity 1. Eigenvectors to the eigenvalue $\lambda = 1$ are in the kernel of $A - 1$ which is the kernel of $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The geometric multiplicity is 1.

RELATION BETWEEN ALGEBRAIC AND GEOMETRIC MULTIPLICITY. (Proof later in the course). The geometric multiplicity is smaller or equal than the algebraic multiplicity.

PRO MEMORIA. Remember that the **geometric mean** \sqrt{ab} of two numbers is smaller or equal to the **algebraic mean** $(a+b)/2$? (This fact is totally* unrelated to the above fact and a mere coincidence of expressions, but it helps to remember it). Quite deeply buried there is a connection in terms of convexity. But this is rather philosophical. .

EXAMPLE. What are the algebraic and geometric multiplicities of $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$?

SOLUTION. The algebraic multiplicity of the eigenvalue 2 is 5. To get the kernel of $A - 2$, one solves the system of equations $x_4 = x_3 = x_2 = x_1 = 0$ so that the geometric multiplicity of the eigenvalues 2 is 4.

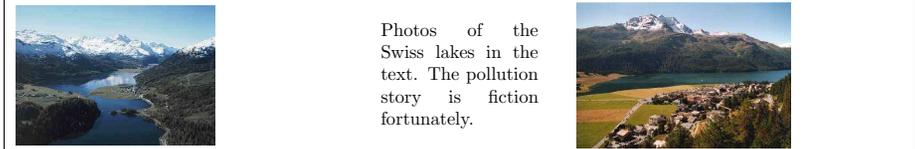
CASE: ALL EIGENVALUES ARE DIFFERENT.

If all eigenvalues are different, then all eigenvectors are linearly independent and all geometric and algebraic multiplicities are 1.
 PROOF. Let λ_i be an eigenvalue different from 0 and assume the eigenvectors are linearly dependent. We have $v_i = \sum_{j \neq i} a_j v_j$ and $\lambda_i v_i = A v_i = A(\sum_{j \neq i} a_j v_j) = \sum_{j \neq i} a_j \lambda_j v_j$ so that $v_i = \sum_{j \neq i} b_j v_j$ with $b_j = a_j \lambda_j / \lambda_i$. If the eigenvalues are different, then $a_j \neq b_j$ and by subtracting $v_i = \sum_{j \neq i} a_j v_j$ from $v_i = \sum_{j \neq i} b_j v_j$, we get $0 = \sum_{j \neq i} (b_j - a_j) v_j = 0$. Now $(n-1)$ eigenvectors of the n eigenvectors are linearly dependent. Use induction.

CONSEQUENCE. If all eigenvalues of a $n \times n$ matrix A are different, there is an **eigenbasis**, a basis consisting of eigenvectors.

EXAMPLE. $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ has eigenvalues 1, 3 to the eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

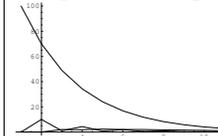
EXAMPLE. (See homework problem 40 in the book).



Photos of the Swiss lakes in the text. The pollution story is fiction fortunately.

The vector $A^n(x)b$ gives the pollution levels in the three lakes (Silvaplana, Sils, St Moritz) after n weeks, where

$$A = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix} \text{ and } b = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} \text{ is the initial pollution.}$$



There is an eigenvector $e_3 = v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to the eigenvalue $\lambda_3 = 0.8$.

There is an eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ to the eigenvalue $\lambda_2 = 0.6$. There is further an eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

to the eigenvalue $\lambda_1 = 0.7$. We know $A^n v_1, A^n v_2$ and $A^n v_3$ explicitly. How do we get the explicit solution $A^n b$? Because $b = 100 \cdot e_1 = 100(v_1 - v_2 + 3v_3)$, we have

$$\begin{aligned} A^n(b) &= 100A^n(v_1 - v_2 + 3v_3) = 100(\lambda_1^n v_1 - \lambda_2^n v_2 + 3\lambda_3^n v_3) \\ &= 100 \left(0.7^n \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + 0.6^n \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 3 \cdot 0.8^n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 100(0.7)^n \\ 100(0.7)^n + 0.6^n \\ 100(-2 \cdot 0.7^n - 0.6^n + 3 \cdot 0.8^n) \end{bmatrix} \end{aligned}$$

DIAGONALIZATION 11/12/2003

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SUMMARY. A $n \times n$ matrix, $A\vec{v} = \lambda\vec{v}$ with eigenvalue λ and eigenvector \vec{v} . The eigenvalues are the roots of the characteristic polynomial $f_A(\lambda) = \det(\lambda - A) = \lambda^n - \text{tr}(A)\lambda^{n-1} + \dots + (-1)^n \det(A)$. The eigenvectors to the eigenvalue λ are in $\ker(\lambda - A)$. The number of times, an eigenvalue λ occurs in the full list of n roots of $f_A(\lambda)$ is called algebraic multiplicity. It is bigger or equal than the geometric multiplicity: $\dim(\ker(\lambda - A))$.

EXAMPLE. The eigenvalues of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\lambda_{\pm} = T/2 \pm \sqrt{T^2/4 - D}$, where $T = a + d$ is the trace and $D = ad - bc$ is the determinant of A . If $c \neq 0$, the eigenvectors are $v_{\pm} = \begin{bmatrix} \lambda_{\pm} - d \\ c \end{bmatrix}$ if $c = 0$, then a, d are eigenvalues to the eigenvectors $\begin{bmatrix} a \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -b \\ a - d \end{bmatrix}$. If $a = d$, then the second eigenvector is parallel to the first and the geometric multiplicity of the eigenvalue $a = d$ is 1.

EIGENBASIS. If A has n different eigenvalues, then A has an eigenbasis, consisting of eigenvectors of A .

DIAGONALIZATION. How does the matrix A look in an eigenbasis? If S is the matrix with the eigenvectors as columns, then we know $\boxed{B = S^{-1}AS}$. We have $S\vec{e}_i = \vec{v}_i$ and $AS\vec{e}_i = \lambda_i\vec{v}_i$ we know $S^{-1}AS\vec{e}_i = \lambda_i\vec{e}_i$. Therefore, B is diagonal with diagonal entries λ_i .

EXAMPLE. $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ has the eigenvalues $\lambda_1 = 2 + \sqrt{3}$ with eigenvector $\vec{v}_1 = [\sqrt{3}, 1]$ and the eigenvalues $\lambda_2 = 2 - \sqrt{3}$ with eigenvector $\vec{v}_2 = [-\sqrt{3}, 1]$. Form $S = \begin{bmatrix} \sqrt{3} & -\sqrt{3} \\ 1 & 1 \end{bmatrix}$ and check $S^{-1}AS = D$ is diagonal.

APPLICATION: FUNCTIONAL CALCULUS. Let A be the matrix in the above example. What is $A^{100} + A^{37} - 1$? The trick is to diagonalize A : $B = S^{-1}AS$, then $B^k = S^{-1}A^kS$ and we can compute $A^{100} + A^{37} - 1 = S(B^{100} + B^{37} - 1)S^{-1}$.

APPLICATION: SOLVING LINEAR SYSTEMS. $x(t+1) = Ax(t)$ has the solution $x(n) = A^n x(0)$. To compute A^n , we diagonalize A and get $x(n) = SB^nS^{-1}x(0)$. This is an explicit formula.

SIMILAR MATRICES HAVE THE SAME EIGENVALUES.

One can see this in two ways:

- 1) If $B = S^{-1}AS$ and \vec{v} is an eigenvector of B to the eigenvalue λ , then $S\vec{v}$ is an eigenvector of A to the eigenvalue λ .
- 2) From $\det(S^{-1}AS) = \det(A)$, we know that the characteristic polynomials $f_B(\lambda) = \det(\lambda - B) = \det(\lambda - S^{-1}AS) = \det(S^{-1}(\lambda - AS)) = \det((\lambda - A)S) = f_A(\lambda)$ are the same.

CONSEQUENCES.

- 1) Because the characteristic polynomials of similar matrices agree, the trace $\text{tr}(A)$ of similar matrices agrees.
- 2) The trace is the sum of the eigenvalues of A . (Compare the trace of A with the trace of the diagonalize matrix.)

THE CAYLEY HAMILTON THEOREM. If A is diagonalizable, then $f_A(A) = 0$.

PROOF. The DIAGONALIZATION $B = S^{-1}AS$ has the eigenvalues in the diagonal. So $f_A(B)$, which contains $f_A(\lambda_i)$ in the diagonal is zero. From $f_A(B) = 0$ we get $Sf_A(B)S^{-1} = f_A(A) = 0$.

By the way: the theorem holds for all matrices: the coefficients of a general matrix can be changed a tiny bit so that all eigenvalues are different. For any such perturbations one has $f_A(A) = 0$. Because the coefficients of $f_A(A)$ depend continuously on A , they are zero 0 in general.

CRITERIA FOR SIMILARITY.

- If A and B have the same characteristic polynomial and diagonalizable, then they are similar.
- If A and B have a different determinant or trace, they are not similar.
- If A has an eigenvalue which is not an eigenvalue of B , then they are not similar.

WHY DO WE WANT TO DIAGONALIZE?

1) **FUNCTIONAL CALCULUS.** If $p(x) = 1 + x + x^2 + x^3/3! + x^4/4!$ be a polynomial and A is a matrix, then $p(A) = 1 + A + A^2/2! + A^3/3! + A^4/4!$ is a matrix. If $B = S^{-1}AS$ is diagonal with diagonal entries λ_i , then $p(B)$ is diagonal with diagonal entries $p(\lambda_i)$. And $p(A) = S p(B)S^{-1}$. This speeds up the calculation because matrix multiplication costs much. The matrix $p(A)$ can be written down with three matrix multiplications, because $p(B)$ is diagonal.

2) **SOLVING LINEAR DIFFERENTIAL EQUATIONS.** A differential equation $\dot{\vec{v}} = A\vec{v}$ is solved by $\vec{v}(t) = e^{At}\vec{v}(0)$, where $e^{At} = 1 + At + A^2t^2/2! + A^3t^3/3! \dots$ is a matrix. (Differentiate this sum with respect to t to get $Ae^{At}\vec{v}(0) = A\vec{v}(t)$.) If we write this in an eigenbasis of A , then $\vec{y}(t) = e^{Bt}\vec{y}(0)$ with the diagonal matrix $B = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$. In other words, we have then explicit solutions $y_j(t) = e^{\lambda_j t} y_j(0)$. Linear differential equations later in this course. It is important motivation.

3) **STOCHASTIC DYNAMICS (i.e MARKOV PROCESSES).** Complicated systems can be modeled by putting probabilities on each possible event and computing the probabilities that an event switches to any other event. This defines a transition matrix. Such a matrix always has an eigenvalue 1. The corresponding eigenvector is the stable probability distribution on the states. If we want to understand, how fast things settle to this equilibrium, we need to know the other eigenvalues and eigenvectors.

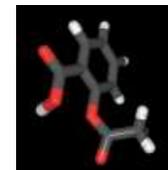
MOLECULAR VIBRATIONS. While quantum mechanics describes the motion of atoms in molecules, the vibrations can be described classically, when treating the atoms as "balls" connected with springs. Such approximations are necessary when dealing with large atoms, where quantum mechanical computations would be too costly. Examples of simple molecules are white phosphorus P_4 , which has tetrahedral shape or methane CH_4 the simplest organic compound or **freon**, CF_2Cl_2 which is used in refrigerants. **Caffeine** or **aspirin** form more complicated molecules.



Freon CF_2Cl_2



Caffeine $C_8H_{10}N_4O_2$



Aspirin $C_9H_8O_4$

WHITE PHOSPHORUS VIBRATIONS. (Differential equations appear later, the context is motivation at this stage). Let x_1, x_2, x_3, x_4 be the positions of the four phosphorus atoms (each of them is a 3-vector). The inter-atomar forces bonding the atoms is modeled by springs. The first atom feels a force $x_2 - x_1 + x_3 - x_1 + x_4 - x_1$ and is accelerated in the same amount. Let's just chose units so that the force is equal to the acceleration. Then

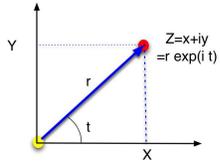
$$\begin{aligned} \ddot{x}_1 &= (x_2 - x_1) + (x_3 - x_1) + (x_4 - x_1) \\ \ddot{x}_2 &= (x_3 - x_2) + (x_4 - x_2) + (x_1 - x_2) \\ \ddot{x}_3 &= (x_4 - x_3) + (x_1 - x_3) + (x_2 - x_3) \\ \ddot{x}_4 &= (x_1 - x_4) + (x_2 - x_4) + (x_3 - x_4) \end{aligned}$$

which has the form $\ddot{x} = Ax$, where the 4×4 matrix



$$A = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are the eigenvectors to the eigenvalues $\lambda_1 = 0, \lambda_2 = -4, \lambda_3 = -4, \lambda_4 = -4$. With $S = [v_1 v_2 v_3 v_4]$, the matrix $B = S^{-1}AS$ is diagonal with entries $0, -4, -4, -4$. The coordinates $y_i = Sx_i$ satisfy $\ddot{y}_1 = 0, \ddot{y}_2 = -4y_2, \ddot{y}_3 = -4y_3, \ddot{y}_4 = -4y_4$ which we can solve y_0 which is the center of mass satisfies $y_0 = a + bt$ (move molecule with constant speed). The motions $y_i = a_i \cos(2t) + b_i \sin(2t)$ of the other eigenvectors are oscillations, called **normal modes**. The general motion of the molecule is a superposition of these modes.



NOTATION. Complex numbers are written as $z = x + iy = r \exp(i\phi) = r \cos(\phi) + ir \sin(\phi)$. The real number $r = |z|$ is called the **absolute value** of z , the value ϕ is the **argument** and denoted by $\arg(z)$. Complex numbers contain the **real numbers** $z = x + i0$ as a subset. One writes $\text{Re}(z) = x$ and $\text{Im}(z) = y$ if $z = x + iy$.

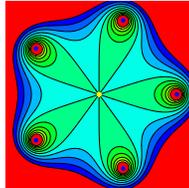
ARITHMETIC. Complex numbers are added like vectors: $x + iy + u + iv = (x + u) + i(y + v)$ and multiplied as $z * w = (x + iy)(u + iv) = xu - yv + i(yu - xv)$. If $z \neq 0$, one can divide $1/z = 1/(x + iy) = (x - iy)/(x^2 + y^2)$.

ABSOLUTE VALUE AND ARGUMENT. The absolute value $|z| = \sqrt{x^2 + y^2}$ satisfies $|zw| = |z| |w|$. The argument satisfies $\arg(zw) = \arg(z) + \arg(w)$. These are direct consequences of the polar representation $z = r \exp(i\phi)$, $w = s \exp(i\psi)$, $zw = rs \exp(i(\phi + \psi))$.

GEOMETRIC INTERPRETATION. If $z = x + iy$ is written as a vector $\begin{bmatrix} x \\ y \end{bmatrix}$, then multiplication with another complex number w is a **dilation-rotation**: a scaling by $|w|$ and a rotation by $\arg(w)$.

THE DE MOIVRE FORMULA. $z^n = \exp(in\phi) = \cos(n\phi) + i \sin(n\phi) = (\cos(\phi) + i \sin(\phi))^n$ follows directly from $z = \exp(i\phi)$ but it is magic: it leads for example to formulas like $\cos(3\phi) = \cos(\phi)^3 - 3 \cos(\phi) \sin^2(\phi)$ which would be more difficult to come by using geometrical or power series arguments. This formula is useful for example in integration problems like $\int \cos(x)^3 dx$, which can be solved by using the above deMoivre formula.

THE UNIT CIRCLE. Complex numbers of length 1 have the form $z = \exp(i\phi)$ and are located on the **unit circle**. The characteristic polynomial $f_A(\lambda) = \lambda^5 - 1$ of the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ has all roots on the unit circle. The roots $\exp(2\pi ki/5)$, for $k = 0, \dots, 4$ lie on the unit circle.



THE LOGARITHM. $\log(z)$ is defined for $z \neq 0$ as $\log|z| + i \arg(z)$. For example, $\log(2i) = \log(2) + i\pi/2$. Riddle: what is i^i ? ($i^i = e^{i \log(i)} = e^{-\pi/2} = e^{-\pi/2}$). The logarithm is not defined at 0 and the imaginary part is define only up to 2π . For example, both $i\pi/2$ and $5i\pi/2$ are equal to $\log(i)$.

HISTORY. The struggle with $\sqrt{-1}$ is historically quite interesting. Nagging questions appeared for example when trying to find closed solutions for roots of polynomials. Cardano (1501-1576) was one of the mathematicians who at least considered complex numbers but called them arithmetic subtleties which were "as refined as useless". With Bombelli (1526-1573), complex numbers found some practical use. Descartes (1596-1650) called roots of negative numbers "imaginary". Although the fundamental theorem of algebra (below) was still not proved in the 18th century, and complex numbers were not fully understood, the square root of minus one $\sqrt{-1}$ was used more and more. Euler (1707-1783) made the observation that $\exp(ix) = \cos x + i \sin x$ which has as a special case the **magic formula** $e^{i\pi} + 1 = 0$ which relate the constants $0, 1, \pi, e$ in one equation.

For decades, many mathematicians still thought complex numbers were a **waste of time**. Others used complex numbers extensively in their work. In 1620, Girard suggested that an equation may have as many roots as its degree in 1620. Leibniz (1646-1716) spent quite a bit of time trying to apply the laws of algebra to complex numbers. He and Johann Bernoulli used imaginary numbers as integration aids. Lambert used complex numbers for map projections, d'Alembert used them in hydrodynamics, while Euler, D'Alembert and Lagrange used them in their incorrect proofs of the fundamental theorem of algebra. Euler write first the symbol i for $\sqrt{-1}$.

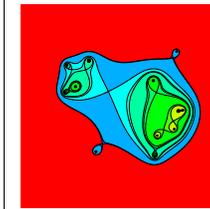
Gauss published the first correct proof of the fundamental theorem of algebra in his doctoral thesis, but still claimed in 1825 that **the true metaphysics of the square root of -1 is elusive** as late as 1825. By 1831 Gauss overcame his uncertainty about complex numbers and published his work on the geometric representation of complex numbers as points in the plane. In 1797, a Norwegian Caspar Wessel (1745-1818) and in 1806 a Swiss clerk named Jean Robert Argand (1768-1822) (who stated the theorem the first time for polynomials with complex coefficients) did similar work. But these efforts went unnoticed. William Rowan Hamilton (1805-1865) (who would also discover the quaternions while walking over a bridge) expressed in 1833 complex numbers as vectors.

Complex numbers continued to develop to **complex function theory** or **chaos theory**, a branch of dynamical systems theory. Complex numbers are helpful in geometry in number theory or in quantum mechanics. Once believed fictitious they are now most "natural numbers" and the "natural numbers" themselves are in fact the most "complex". A philosopher who asks "does $\sqrt{-1}$ really exist?" might be shown the representation of $x + iy$ as $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$. When adding or multiplying such dilation-rotation matrices, they behave like complex numbers: for example $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ plays the role of i .

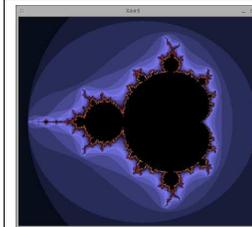
FUNDAMENTAL THEOREM OF ALGEBRA. (Gauss 1799) A polynomial of degree n has exactly n roots.

CONSEQUENCE: A $n \times n$ MATRIX HAS n EIGENVALUES. The characteristic polynomial $f_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$ satisfies $f_A(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$, where λ_i are the roots of f .

TRACE AND DETERMINANT. Comparing $f_A(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$ with $\lambda^n - \text{tr}(A)\lambda + \dots + (-1)^n \det(A)$ gives $\text{tr}(A) = \lambda_1 + \dots + \lambda_n$, $\det(A) = \lambda_1 \dots \lambda_n$.



COMPLEX FUNCTIONS. The characteristic polynomial is an example of a function f from \mathbb{C} to \mathbb{C} . The graph of this function would live in $\mathbb{C} \times \mathbb{C}$ which corresponds to a four dimensional real space. One can visualize the function however with the real-valued function $z \mapsto |f(z)|$. The figure to the left shows the contour lines of such a function $z \mapsto |f(z)|$, where f is a polynomial.



ITERATION OF POLYNOMIALS. A topic which is off this course (it would be a course by itself) is the iteration of polynomials like $f_c(z) = z^2 + c$. The set of parameter values c for which the iterates $f_c(0), f_c^2(0) = f_c(f_c(0)), \dots, f_c^n(0)$ stay bounded is called the **Mandelbrot set**. It is the fractal black region in the picture to the left. The now already dusty object appears everywhere, from photoshop plugins to decorations. In Mathematica, you can compute the set very quickly (see <http://www.math.harvard.edu/computing/math/mandelbrot.m>).

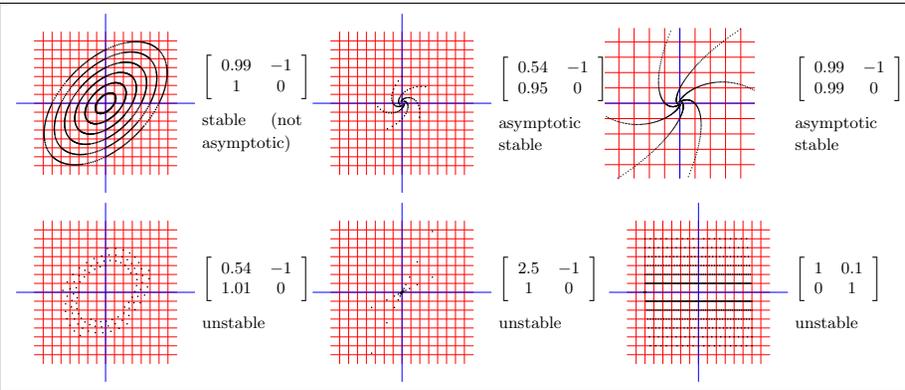
COMPLEX NUMBERS IN MATHEMATICA OR MAPLE. In both computer algebra systems, the letter I is used for $i = \sqrt{-1}$. In Maple, you can ask $\log(1 + I)$, in Mathematica, this would be $\text{Log}[1 + I]$. Eigenvalues or eigenvectors of a matrix will in general involve complex numbers. For example, in Mathematica, $\text{Eigenvalues}[A]$ gives the eigenvalues of a matrix A and $\text{Eigensystem}[A]$ gives the eigenvalues and the corresponding eigenvectors.

EIGENVALUES AND EIGENVECTORS OF A ROTATION. The rotation matrix $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$ has the characteristic polynomial $\lambda^2 - 2 \cos(\phi) \lambda + 1$. The eigenvalues are $\cos(\phi) \pm \sqrt{\cos^2(\phi) - 1} = \cos(\phi) \pm i \sin(\phi) = \exp(\pm i\phi)$. The eigenvector to $\lambda_1 = \exp(i\phi)$ is $v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and the eigenvector to $\lambda_2 = \exp(-i\phi)$ is $v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$.

LINEAR DYNAMICAL SYSTEM. A linear map $x \mapsto Ax$ defines a **dynamical system**. Iterating the map produces an **orbit** $x_0, x_1 = Ax, x_2 = A^2 = AAx, \dots$. The vector $x_n = A^n x_0$ describes the situation of the system at **time** n .

Where does x_n go when time evolves? Can one describe what happens asymptotically when time n goes to infinity?

In the case of the Fibonacci sequence x_n which gives the number of rabbits in a rabbit population at time n , the population grows essentially exponentially. Such a behavior would be called **unstable**. On the other hand, if A is a rotation, then $A^n \vec{v}$ stays bounded which is a type of **stability**. If A is a dilation with a dilation factor < 1 , then $A^n \vec{v} \rightarrow 0$ for all \vec{v} , a thing which we will call **asymptotic stability**. The next pictures show experiments with some **orbits** $A^n \vec{v}$ with different matrices.



ASYMPTOTIC STABILITY. The origin $\vec{0}$ is invariant under a linear map $T(\vec{x}) = A\vec{x}$. It is called **asymptotically stable** if $A^n(\vec{x}) \rightarrow \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$.

EXAMPLE. Let $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ be a dilation rotation matrix. Because multiplication with such a matrix is analogue to the multiplication with a complex number $z = p + iq$, the matrix A^n corresponds to a multiplication with $(p + iq)^n$. Since $|(p + iq)^n| = |p + iq|^n$, the origin is asymptotically stable if and only if $|p + iq| < 1$. Because $\det(A) = |p + iq|^2 = |z|^2$, rotation-dilation matrices A have an asymptotically stable origin if and only if $|\det(A)| < 1$. Dilation-rotation matrices $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ have eigenvalues $p \pm iq$ and can be diagonalized in the complex.

EXAMPLE. If a matrix A has an eigenvalue $|\lambda| \geq 1$ to an eigenvector \vec{v} , then $A^n \vec{v} = \lambda^n \vec{v}$, whose length is $|\lambda|^n$ times the length of \vec{v} . So, we have no asymptotic stability if an eigenvalue satisfies $|\lambda| \geq 1$.

STABILITY. The book also writes "stable" for "asymptotically stable". This is ok to abbreviate. Note however that the commonly used term "stable" also includes linear maps like rotations, reflections or the identity. It is therefore preferable to leave the attribute "asymptotic" in front of "stable".

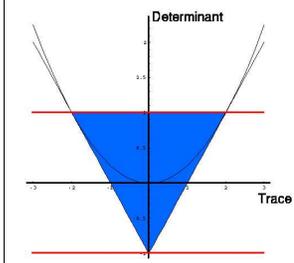
ROTATIONS. Rotations $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$ have the eigenvalue $\exp(\pm i\phi) = \cos(\phi) + i \sin(\phi)$ and are not asymptotically stable.

DILATIONS. Dilations $\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ have the eigenvalue r with algebraic and geometric multiplicity 2. Dilations are asymptotically stable if $|r| < 1$.

CRITERION. A linear dynamical system $x \mapsto Ax$ has an asymptotically stable origin if and only if all its eigenvalues have an absolute value < 1 .

PROOF. We have already seen in Example 3, that if one eigenvalue satisfies $|\lambda| > 1$, then the origin is not asymptotically stable. If $|\lambda_i| < 1$ for all i and all eigenvalues are different, there is an eigenbasis v_1, \dots, v_n . Every x can be written as $x = \sum_{j=1}^n x_j v_j$. Then, $A^n x = A^n (\sum_{j=1}^n x_j v_j) = \sum_{j=1}^n x_j \lambda_j^n v_j$ and because $|\lambda_j|^n \rightarrow 0$, there is stability. The proof of the general (nondiagonalizable) case will be accessible later.

THE 2-DIMENSIONAL CASE. The characteristic polynomial of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$. If $c \neq 0$, the eigenvalues are $\lambda_{\pm} = \text{tr}(A)/2 \pm \sqrt{(\text{tr}(A)/2)^2 - \det(A)}$. If the **discriminant** $(\text{tr}(A)/2)^2 - \det(A)$ is nonnegative, then the eigenvalues are real. This happens below the parabola, where the discriminant is zero.



CRITERION. In two dimensions we have asymptotic stability if and only if $(\text{tr}(A), \det(A))$ is contained in the **stability triangle** bounded by the lines $\det(A) = 1, \det(A) = \text{tr}(A) - 1$ and $\det(A) = -\text{tr}(A) - 1$.

PROOF. Write $T = \text{tr}(A)/2, D = \det(A)$. If $|D| \geq 1$, there is no asymptotic stability. If $\lambda = T + \sqrt{T^2 - D} = \pm 1$, then $T^2 - D = (\pm 1 - T)^2$ and $D = 1 \pm 2T$. For $D \leq -1 + |2T|$ we have a real eigenvalue ≥ 1 . The conditions for stability is therefore $D > |2T| - 1$. It implies automatically $D > -1$ so that the triangle can be described shortly as $|\text{tr}(A)| - 1 < \det(A) < 1$.

EXAMPLES.

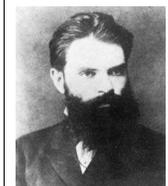
- 1) The matrix $A = \begin{bmatrix} 1 & 1/2 \\ -1/2 & 1 \end{bmatrix}$ has determinant $5/4$ and trace 2 and the origin is unstable. It is a dilation-rotation matrix which corresponds to the complex number $1 + i/2$ which has an absolute value > 1 .
- 2) A rotation A is never asymptotically stable: $\det(A) = 1$ and $\text{tr}(A) = 2 \cos(\phi)$. Rotations are the upper side of the **stability triangle**.
- 3) A dilation is asymptotically stable if and only if the scaling factor has norm < 1 .
- 4) If $\det(A) = 1$ and $\text{tr}(A) < 2$ then the eigenvalues are on the unit circle and there is no asymptotic stability.
- 5) If $\det(A) = -1$ (like for example Fibonacci) there is no asymptotic stability. For $\text{tr}(A) = 0$, we are a corner of the stability triangle and the map is a reflection, which is not asymptotically stable neither.

SOME PROBLEMS.

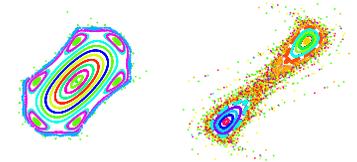
- 1) If A is a matrix with asymptotically stable origin, what is the stability of 0 with respect to A^T ?
- 2) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to A^{-1} ?
- 3) If A is a matrix which has an asymptotically stable origin, what is the stability with respect to A^{100} ?

ON THE STABILITY QUESTION.

For general dynamical systems, the question of stability can be very difficult. We deal here only with linear dynamical systems, where the eigenvalues determine everything. For nonlinear systems, the story is not so simple even for simple maps like the Henon map. The questions go deeper: it is for example not known, whether our solar system is stable. We don't know whether in some future, one of the planets could get expelled from the solar system (this is a mathematical question because the escape time would be larger than the life time of the sun). For other dynamical systems like the atmosphere of the earth or the stock market, we would really like to know what happens in the near future ...



A pioneer in stability theory was Aleksandr Lyapunov (1857-1918). For nonlinear systems like $x_{n+1} = gx_n - x_n^3 - x_{n-1}$ the stability of the origin is nontrivial. As with Fibonacci, this can be written as $(x_{n+1}, x_n) = (gx_n - x_n^2 - x_{n-1}, x_n) = A(x_n, x_{n-1})$ called **cubic Henon map** in the plane. To the right are orbits in the cases $g = 1.5, g = 2.5$.



The first case is stable (but proving this requires a fancy theory called KAM theory), the second case is unstable (in this case actually the linearization at $\vec{0}$ determines the picture).

CHECKLIST SECOND MIDTERM, 11/19/2003

Math21b, O.Knill

The EIGENVECTORS AND EIGENVALUES of a matrix A reveal the structure of A . Diagonalization in general eases the computations with A . It allows to find explicit formulas for LINEAR DYNAMICAL SYSTEMS $x \mapsto Ax$. Such systems are important for example in probability theory. The dot product leads to the notion of ORTHOGONALITY, allows measurements of angles and lengths and leads to geometric notations like rotation, reflection or projection in arbitrary dimensions. Least square solutions of $Ax = b$ allow for example to solve fitting problems. DETERMINANTS of matrices appear in the definition of the characteristic polynomial and as volumes of parallelepipeds or as scaling values in change of variables. The notion allows to give explicit formulas for the inverse of a matrix or to solutions of $Ax = b$.

ORTHOGONAL $\vec{v} \cdot \vec{w} = 0$.
LENGTH $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.
UNIT VECTOR \vec{v} with $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = 1$.
ORTHOGONAL SET v_1, \dots, v_n : pairwise orthogonal.
ORTHONORMAL SET orthogonal and length 1.
ORTHONORMAL BASIS A basis which is orthonormal.
ORTHOGONAL TO V v is orthogonal to V if $v \cdot x = 0$ for all $x \in V$.
ORTHOGONAL COMPLEMENT OF V Linear space $V^\perp = \{v \mid v \text{ orthogonal to } V\}$.
PROJECTION ONTO V orth. basis v_1, \dots, v_n in V , $\text{perp}_V(x) = (v_1 \cdot x)v_1 + \dots + (v_n \cdot x)v_n$.
GRAMM-SCHMIDT Recursive $u_i = v_i - \text{proj}_{V_{i-1}} v_i$, $w_i = u_i / \|u_i\|$ leads to orthonormal basis.
QR-FACTORIZATION $Q = [w_1 \dots w_n]$, $R_{ii} = u_i$, $[R]_{ij} = w_i \cdot v_j, j > i$.
TRANSPOSE $[A^T]_{ij} = A_{ji}$. Transposition switches rows and columns.
SYMMETRIC $A^T = A$.
SKESYMMETRIC $A^T = -A$ ($\Rightarrow R = e^A$ orthogonal: $R^T = e^{A^T} = e^{-A} = R^{-1}$).
DOT PRODUCT AS MATRIX PRODUCT $v \cdot w = v^T \cdot w$.
ORTHOGONAL MATRIX $Q^T Q = 1$.
ORTHOGONAL PROJECTION onto V is AA^T , columns \vec{v}_i are orthonormal basis in V .
ORTHOGONAL PROJECTION onto V is $A(A^T A)^{-1} A^T$, columns \vec{v}_i are basis in V .
NORMAL EQUATION to $Ax = b$ is the consistent system $A^T Ax = A^T b$.
LEAST SQUARE SOLUTION of $A\vec{x} = \vec{b}$ is $\vec{x}_* = (A^T A)^{-1} A^T \vec{b}$.
ANGLE between two vectors x, y is $\alpha = \arccos((x \cdot y) / (\|x\| \|y\|))$.
CORRELATION COEFFICIENT $(x \cdot y) / (\|x\| \|y\|)$ is $\cos(\text{angle})$.
SCATTER PLOT Visualization of data points (x_i, y_i) in the plane.
DETERMINANT $\det(A) = (\sum_{\text{even } \pi} \pi - \sum_{\text{odd } \pi} \pi) A_{1\pi(1)} A_{2\pi(2)} \dots A_{n\pi(n)}$.
PARALLELEPIPED Image of unit cube by A . Spanned by columns of A , volume $\sqrt{\det(A^T A)}$.
MINOR A_{ij} , the matrix with row i and column j deleted.
CLASSICAL ADJOINT $\text{adj}(A)_{ij} = (-1)^{i+j} \det(A_{ji})$ (note switch of ij).
ORIENTATION $\text{sign}(\det(A))$ defines orientantation of column vectors of A .
TRACE is $\text{tr}(A) = \sum_i A_{ii}$, the sum of diagonal elements of A .
CHARACTERISTIC POLYNOMIAL $f_A(\lambda) = \det(\lambda - A) = \lambda^n - \text{tr}(A)\lambda + \dots + (-1)^n \det(A)$.
EIGENVALUES AND EIGENVECTORS $Av = \lambda v, v \neq 0$, eigenvalue λ , eigenvector v .
FACTORISATION OF $f_A(\lambda)$ Have $f_A(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$ with roots λ_i .
ALGEBRAIC MULTIPLICITY k If $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$ with $g(\lambda_0) \neq 0$.
GEOMETRIC MULTIPLICITY The dimension of the kernel of $\lambda - A$.
KERNEL AND EIGENVECTORS Vectors in the kernel of A are eigenvectors of A .
EIGENBASIS Basis which consists of eigenvectors of A .
COMPLEX NUMBERS $z = x + iy = |z| \exp(i \arg(z)) = r e^{i\phi} = r \exp(i\phi) = r \cos(\phi) + ir \sin(\phi)$.
MODULUS AND ARGUMENT $|z| = |x + iy| = \sqrt{x^2 + y^2}$, $\phi = \arg(z) = \arctan(y/x)$.
CONJUGATE $\bar{z} = x - iy$ if $z = x + iy$.
LINEAR DYNAMICAL SYSTEM Linear map $x \mapsto Ax$ defines orbit $\vec{x}(t+1) = A\vec{x}(t)$.

Skills checklist

COMPUTE DETERMINANTS IN DIFFERENT WAYS (row reduction, Laplace, volume, patterns).
 GRAMM-SCHMIDT ORTHOGONALISATION (algorithm).
 COMPUTING EIGENVALUES OF A (factoring characteristic polynomial).
 COMPUTING EIGENVECTORS OF A (determining kernel of $\lambda - A$).
 COMPUTING ALGEBRAIC AND GEOMETRIC MULTIPLICITIES (know the definitions).
 COMPUTING ORTHOGONAL PROJECTION ONTO LINEAR SUBSPACES (formula).
 PRODUCE LEAST SQUARE SOLUTION OF LINEAR EQUATION (formula).
 ALGEBRA OF MATRICES (multiplication, inverse, recall rank, image, kernel, inverse).
 CALCULATION WITH COMPLEX NUMBERS (operations, roots, identifying $\text{Re}(z), \text{Im}(z)$).
 DIAGONALIZE MATRIX (find eigensystem, build S so that $S^{-1}AS$ is diagonal).
 DETERMINE STABILITY OF A LINEAR SYSTEM (eigenvalue criterion, especially 2D case).

ORTHOGONAL IMPLIES INDEPENDENT. Orthogonal vectors are linearly independent.
 ORTHOGONAL PLUS SPAN IMPLIES BASIS. n orthogonal vectors \mathbf{R}^n form a basis.
 ORTHOGONAL COMPLEMENT IS LINEAR SPACE. Notation V^\perp . Have $(V^\perp)^\perp = V$.
 LINE TO PROJECTION IS ORTHOGONAL TO V . $\vec{x} - \text{proj}_V(\vec{x})$ is orthogonal to V .
 PYTHAGORAS: x, y orthogonal $\Rightarrow \|x + y\|^2 = \|x\|^2 + \|y\|^2$.
 PROJECTION CONTRACTS $\|\text{proj}_V(\vec{x})\| \leq \|\vec{x}\|$.
 IMAGE OF A IS ORTHOGONAL COMPLEMENT TO KERNEL OF A^T .
 DIMENSION OF ORTHOGONAL COMPLEMENT. $\dim(V) + \dim(V^\perp) = n$.
 CAUCHY-SCHWARTZ INEQUALITY: $|x \cdot y| \leq \|x\| \|y\|$.
 TRIANGLE INEQUALITY: $\|x + y\| \leq \|x\| + \|y\|$.
 ROW VECTORS OF A are orthogonal to $\ker(A)$. Short $\text{im}(A^T) = \ker(A)^\perp$.
 ORTHOGONAL TRANSFORMATIONS preserve angle, length. Columns form orthonormal basis.
 ORTHOGONAL PROJECTION: $P = A(A^T A)^{-1} A^T$ onto V with $A = [v_1, \dots, v_n], V = \text{im}(A)$.
 ORTHOGONAL PROJECTION: onto V is AA^T if $A = [v_1, \dots, v_n]$ is orthogonal.
 ORTHOGONAL PROJECTIONS are not orthogonal transformations in general.
 KERNEL OF A AND $A^T A$ are the same: $\ker(A) = \ker(A^T A)$.
 DETERMINANTS $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$. $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - fha - bdi$.
 DETERMINANT OF DIAGONAL OR TRIANGULAR MATRIX: product of diagonal entries.
 DETERMINANT OF BLOCK MATRIX: $\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A)\det(B)$.
 PROPERTIES OF DETERMINANTS. $\det(AB) = \det(A)\det(B)$, $\det(A^{-1}) = 1/\det(A)$.
 PROPERTIES OF DETERMINANTS. $\det(SAS^{-1}) = \det(A)$, $\det(A^T) = \det(A)$.
 LINEARITY OF DETERMINANT. $\det([A\vec{a}B]) + \det([A\vec{b}B]) = \det([A(\vec{a} + \vec{b})B])$, $\det([A\lambda\vec{a}B]) = \lambda \det([A\vec{a}B])$.
 DETERMINANT OF $[A]_{ij} = \vec{v}_i \cdot \vec{v}_j$ is square of determinant of $B = [\vec{v}_1, \dots, \vec{v}_n]$.
 SWITCHING OF TWO ROWS. $\det(B) = -\det(A)$.
 ADDING ROW TO GIVEN DIFFERENT ROW: $\det(B) = \det(A)$
 PARALLELEPIPED. $|\det(A)| = \text{vol}(E)$ with E =parallelepiped spanned by columns of A .
 K-EPIPED. $\sqrt{\det(A^T A)} = \text{vol}(k\text{-dimensional parallelepiped spanned by column vectors of } A)$.
 RREF. $\det(A) = (-1)^s (\prod_i c_i) \det(\text{rref}(A))$ with c_i row scaling factors and s row switches.
 IN ODD DIMENSIONS a real matrix has a real eigenvalue.
 IN EVEN DIMENSIONS a real matrix with negative determinant has real eigenvalue.
 PROPERTIES OF TRANSPOSE. $(A^T)^T = A$, $(AB)^T = B^T A^T$, $(A^{-1})^T = (A^T)^{-1}$.
 DIAGONALISATION. $A \times n \times n$ matrix, S eigenvectors of A in columns, $S^{-1}AS$ diagonal.
 JORDAN NORMAL FORM: In the complex, every A can be brought into Jordan normal form.
 NONTRIVIAL KERNEL $\Leftrightarrow \det(A) = 0$.
 INVERTIBLE MATRIX $\Leftrightarrow \det(A) \neq 0$.
 LAPLACE EXPANSION. $\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + \dots + (-1)^{i+n} a_{in} \det(A_{in})$
 ORTHOGONAL MATRICES A have $\det(A) = \pm 1$
 ROTATIONS satisfy $\det(A) = 1$ in all dimensions.
 ROTATIONS with angle ϕ in the plane have eigenvalues $\exp(i\phi)$.
 QR DECOMPOSITION. $A = QR$ orthogonal A , upper triangular R . Have $|\det(A)| = \prod_{i=1}^n R_{ii}$.
 CRAMER'S RULE. Solve $Ax = b$ by $x_i = \det(A_i) / \det(A)$, where A_i is A with b in column i .
 CLASSICAL ADJOINT AND INVERSE. $A^{-1} = \text{adj}(A) / \det(A)$.
 DETERMINANT IS PRODUCT OF EIGENVALUES. $\det(A) = \prod_i \lambda_i$.
 TRACE IS SUM OF EIGENVALUES. $\text{tr}(A) = \sum_i \lambda_i$.
 GEOMETRIC MULTIPLICITY \leq ALGEBRAIC MULTIPLICITY.
 DIFFERENT EIGENVALUES \Rightarrow EIGENSYSTEM. $\lambda_i \neq \lambda_j, i \neq j \Rightarrow$ eigenvectors form basis.
 EIGENVALUES OF A^T agree with eigenvalues of A (same characteristic polynomial).
 RANK OF A^T is equal to the rank of A .
 REFLECTION at linear k -dimensional subspace in \mathbf{R}^n has determinant $(-1)^{(n-k)}$.
 DE MOIVRE FORMULA: $z^n = \exp(in\phi) = \cos(n\phi) + i \sin(n\phi) = (\cos(\phi) + i \sin(\phi))^n$.
 FUNDAMENTAL THEOREM OF ALGEBRA. $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + \lambda_0$ has n roots.
 NUMBER OF EIGENVALUES. $A \times n \times n$ matrix has exactly n eigenvalues (count multiplicity).
 POWER OF A MATRIX. A^n has eigenvalues λ^n if A has eigenvalue λ .
 EIGENVALUES OF $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are $\lambda_{\pm} = \text{tr}(A)/2 \pm \sqrt{(\text{tr}(A)/2)^2 - \det(A)}$.
 EIGENVECTORS OF $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $c \neq 0$ are $v_{\pm} = [\lambda_{\pm} - d, c]$.
 ROTATION-DILATION MATRIX: $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$, eigenvalues $p \pm iq$, eigenvectors $(\pm i, 1)$.
 ROTATION-DILATION MATRIX: linear stable origin if and only if $|\det(A)| < 1$.
 DATA FITTING: Least square solution of $A\vec{x} = \vec{b}$, where A and b depend on data and functions.

SYMMETRIC MATRICES 11/21/2003

Math 21b, O. Knill

SYMMETRIC MATRICES. A matrix A with real entries is **symmetric**, if $A^T = A$.

EXAMPLES. $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ is symmetric, $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ is not symmetric.

EIGENVALUES OF SYMMETRIC MATRICES. Symmetric matrices A have real eigenvalues.

PROOF. The dot product is extend to complex vectors as $(v, w) = \sum_i \bar{v}_i w_i$. For real vectors it satisfies $(v, w) = v \cdot w$ and has the property $(Av, w) = (v, A^T w)$ for real matrices A and $(\lambda v, w) = \bar{\lambda}(v, w)$ as well as $(v, \lambda w) = \lambda(v, w)$. Now $\bar{\lambda}(v, v) = (\lambda v, v) = (Av, v) = (v, A^T v) = (v, Av) = (v, \lambda v) = \lambda(v, v)$ shows that $\bar{\lambda} = \lambda$ because $(v, v) \neq 0$ for $v \neq 0$.

EXAMPLE. $A = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ has eigenvalues $p + iq$ which are real if and only if $q = 0$.

EIGENVECTORS OF SYMMETRIC MATRICES. Symmetric matrices have an orthonormal eigenbasis

PROOF. If $Av = \lambda v$ and $Aw = \mu w$. The relation $\lambda(v, w) = (\lambda v, w) = (Av, w) = (v, A^T w) = (v, Aw) = (v, \mu w) = \mu(v, w)$ is only possible if $(v, w) = 0$ if $\lambda \neq \mu$.

WHY ARE SYMMETRIC MATRICES IMPORTANT? In applications, matrices are often symmetric. For example in **geometry** as **generalized dot products** $v \cdot Av$, or in **statistics** as **correlation matrices** $\text{Cov}[X_k, X_l]$ or in quantum mechanics as **observables** or in **neural networks** as **learning maps** $x \mapsto \text{sign}(Wx)$ or in graph theory as **adjacency matrices** etc. etc. Symmetric matrices play the same role as real numbers do among the complex numbers. Their eigenvalues often have physical or geometrical interpretations. One can also calculate with symmetric matrices like with numbers: for example, we can solve $B^2 = A$ for B if A is symmetric matrix and B is square root of A .) This is not possible in general: try to find a matrix B such that $B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dots$

RECALL. We have seen when an eigenbasis exists, a matrix A can be transformed to the diagonal matrix $B = S^{-1}AS$, where $S = [v_1, \dots, v_n]$. The matrices A and B are **similar**. B is called the **diagonalization** of A . Similar matrices have the same characteristic polynomial $\det(B - \lambda) = \det(S^{-1}(A - \lambda)S) = \det(A - \lambda)$ and have therefore the same determinant, trace and eigenvalues. Physicists call the set of eigenvalues also **the spectrum**. They say that these matrices are isospectral. The spectrum is what you "see" (etymologically the name origins from the fact that in quantum mechanics the spectrum of radiation can be associated with eigenvalues of matrices.)

SPECTRAL THEOREM. Symmetric matrices A can be diagonalized $B = S^{-1}AS$ with an orthogonal S .

PROOF. If all eigenvalues are different, there is an eigenbasis and diagonalization is possible. The eigenvectors are all orthogonal and $B = S^{-1}AS$ is diagonal containing the eigenvalues. In general, we can change the matrix A to $A = A + (C - A)t$ where C is a matrix with pairwise different eigenvalues. Then the eigenvalues are different for all except finitely many t . The orthogonal matrices S_t converges for $t \rightarrow 0$ to an orthogonal matrix S and S diagonalizes A .

WAIT A SECOND ... Why could we not perturb a general matrix A_t to have disjoint eigenvalues and A_t could be diagonalized: $S_t^{-1}A_t S_t = B_t$? The problem is that S_t might become singular for $t \rightarrow 0$. See problem 5) first practice exam.

EXAMPLE 1. The matrix $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ has the eigenvalues $a + b, a - b$ and the eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} / \sqrt{2}$. They are orthogonal. The orthogonal matrix $S = [v_1 \ v_2]$ diagonalized A .

EXAMPLE 2. The 3×3 matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has 2 eigenvalues 0 to the eigenvectors $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ and one eigenvalue 3 to the eigenvector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. All these vectors can be made orthogonal and a diagonalization is possible even so the eigenvalues have multiplicities.

SQUARE ROOT OF A MATRIX. How do we find a square root of a given symmetric matrix? Because $S^{-1}AS = B$ is diagonal and we know how to take a square root of the diagonal matrix B , we can form $C = S\sqrt{B}S^{-1}$ which satisfies $C^2 = S\sqrt{B}S^{-1}S\sqrt{B}S^{-1} = SBS^{-1} = A$.

RAYLEIGH FORMULA. We write also $(\vec{v}, \vec{w}) = \vec{v} \cdot \vec{w}$. If $\vec{v}(t)$ is an eigenvector of length 1 to the eigenvalue $\lambda(t)$ of a symmetric matrix $A(t)$ which depends on t , differentiation of $(A(t) - \lambda(t))\vec{v}(t) = 0$ with respect to t gives $(A' - \lambda')v + (A - \lambda)v' = 0$. The symmetry of $A - \lambda$ implies $0 = (v, (A' - \lambda')v) + (v, (A - \lambda)v') = (v, (A' - \lambda')v)$. We see that the **Rayleigh quotient** $\lambda' = (A'v, v)$ is a polynomial in t if $A(t)$ only involves terms t, t^2, \dots, t^m . The formula shows how $\lambda(t)$ changes, when t varies. For example, $A(t) = \begin{bmatrix} 1 & t^2 \\ t^2 & 1 \end{bmatrix}$ has for $t = 2$ the eigenvector $\vec{v} = [1, 1]/\sqrt{2}$ to the eigenvalue $\lambda = 5$. The formula tells that $\lambda'(2) = (A'(2)\vec{v}, \vec{v}) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \vec{v}, \vec{v} = 4$. Indeed, $\lambda(t) = 1 + t^2$ has at $t = 2$ the derivative $2t = 4$.

EXHIBITION. "Where do symmetric matrices occur?" Some informal motivation:

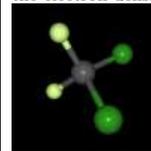
I) PHYSICS: In **quantum mechanics** a system is described with a vector $v(t)$ which depends on time t . The evolution is given by the **Schrodinger equation** $\dot{v} = i\hbar Lv$, where L is a symmetric matrix and \hbar is a small number called the Planck constant. As for any linear differential equation, one has $v(t) = e^{i\hbar Lt}v(0)$. If $v(0)$ is an eigenvector to the eigenvalue λ , then $v(t) = e^{i\hbar \lambda t}v(0)$. Physical observables are given by symmetric matrices too. The matrix L represents the energy. Given $v(t)$, the value of the observable $A(t)$ is $v(t) \cdot Av(t)$. For example, if v is an eigenvector to an eigenvalue λ of the energy matrix L , then the energy of $v(t)$ is λ .



This is called the **Heisenberg picture**. In order that $v \cdot A(t)v = v(t) \cdot Av(t) = S(t)v \cdot AS(t)v$ we have $A(t) = S(T)^*AS(t)$, where $S^* = \overline{S^T}$ is the correct generalization of the adjoint to complex matrices. $S(t)$ satisfies $S(t)^*S(t) = 1$ which is called **unitary** and the complex analogue of orthogonal. The matrix $A(t) = S(t)^*AS(t)$ has the same eigenvalues as A and is **similar** to A .

II) CHEMISTRY. The **adjacency matrix** A of a graph with n vertices determines the graph: one has $A_{ij} = 1$ if the two vertices i, j are connected and zero otherwise. The matrix A is symmetric. The eigenvalues λ_j are real and can be used to analyze the graph. One interesting question is to what extent the eigenvalues determine the graph.

In chemistry, one is interested in such problems because it allows to make rough computations of the electron density distribution of molecules. In this so called **Hückel theory**, the molecule is represented as a graph. The eigenvalues λ_j of that graph approximate the energies an electron on the molecule. The eigenvectors describe the electron density distribution.



The **Freon molecule** for example has 5 atoms. The adjacency matrix is

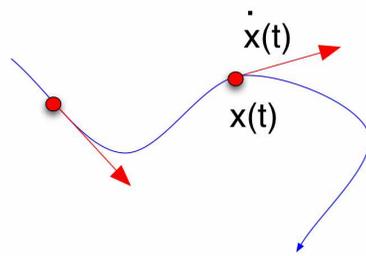
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix A has the eigenvalue 0 with multiplicity 3 ($\ker(A)$ is obtained immediately from the fact that 4 rows are the same) and the eigenvalues 2, -2. The eigenvector to the eigenvalue ± 2 is $[\pm 2 \ 1 \ 1 \ 1 \ 1]^T$.

III) STATISTICS. If we have a random vector $X = [X_1, \dots, X_n]$ and $E[X_k]$ denotes the expected value of X_k , then $[A]_{kl} = E[(X_k - E[X_k])(X_l - E[X_l])] = E[X_k X_l] - E[X_k]E[X_l]$ is called the **covariance matrix** of the random vector X . It is a symmetric $n \times n$ matrix. Diagonalizing this matrix $B = S^{-1}AS$ produces new random variables which are **uncorrelated**.

For example, if X is the sum of two dice and Y is the value of the second dice then $E[X] = [(1+1) + (1+2) + \dots + (6+6)]/36 = 7$, you throw in average a sum of 7 and $E[Y] = (1+2 + \dots + 6)/6 = 7/2$. The matrix entry $A_{11} = E[X^2] - E[X]^2 = [(1+1) + (1+2) + \dots + (6+6)]/36 - 7^2 = 35/6$ known as the **variance** of X , and $A_{22} = E[Y^2] - E[Y]^2 = (1^2 + 2^2 + \dots + 6^2)/6 - (7/2)^2 = 35/12$ known as the **variance** of Y and $A_{12} = E[XY] - E[X]E[Y] = 35/12$. The covariance matrix is the symmetric matrix $A = \begin{bmatrix} 35/6 & 35/12 \\ 35/12 & 35/12 \end{bmatrix}$.

CONTINUOUS DYNAMICAL SYSTEMS. A differential equation $\frac{d}{dt}\vec{x} = f(\vec{x})$ defines a dynamical system. The solutions is a curve $\vec{x}(t)$ which has the velocity vector $f(\vec{x}(t))$ for all t . We often write \dot{x} for $\frac{d}{dt}x$.

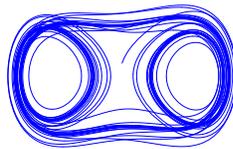


ONE DIMENSION. A system $\dot{x} = g(x, t)$ (written in the form $\dot{x} = g(x, t), \dot{t} = 1$) and often has explicit solutions

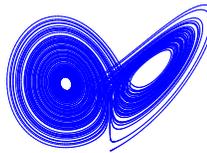
- If $\dot{x} = g(t)$, then $x(t) = \int_0^t g(t) dt$.
- If $\dot{x} = h(x)$, then $dx/h(x) = dt$ and so $t = \int_0^x dx/h(x) = H(x)$ so that $x(t) = H^{-1}(t)$.
- If $\dot{x} = g(t)/h(x)$, then $H(x) = \int_0^x h(x) dx = \int_0^t g(t) dt = G(t)$ so that $x(t) = H^{-1}(G(t))$.

In general, we have no closed form solutions in terms of known functions. The solution $x(t) = \int_0^t e^{-t^2} dt$ of $\dot{x} = e^{-t^2}$ for example can not be expressed in terms of functions exp, sin, log, $\sqrt{\cdot}$ etc but it can be solved using Taylor series: because $e^{-t^2} = 1 - t^2 + t^4/2! - t^6/3! + \dots$ taking coefficient wise the anti-derivatives gives: $x(t) = t - t^3/3 + t^4/(32!) - t^7/(73!) + \dots$

HIGHER DIMENSIONS. In higher dimensions, **chaos can set in** and the dynamical system can become unpredictable.



The nonlinear **Lorenz system** to the right $\dot{x}(t) = 10(y(t) - x(t)), \dot{y}(t) = -x(t)z(t) + 28x(t) - y(t), \dot{z}(t) = x(t) * y(t) - 8z(t)/3$ shows a "strange attractor". Evenso completely **deterministic**: (from $x(0)$ all the path $x(t)$ is determined), there are observables which can be used as a random number generator. The **Duffing system** $\ddot{x} + \dot{x}.10 - x + x^3 - 12 \cos(t) = 0$ to the left can be written in the form $\dot{v} = f(v)$ with a vector $v = (x, \dot{x}, t)$.

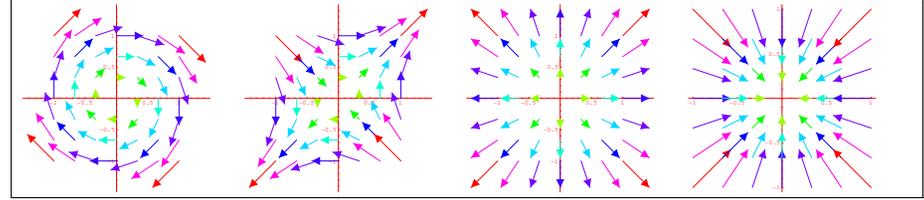


1D LINEAR DIFFERENTIAL EQUATIONS. A linear differential equation in one dimension $\dot{x} = \lambda x$ has the solution is $x(t) = e^{\lambda t} x(0)$. This differential equation appears

- as **population models** with $\lambda > 0$: birth rate of the popolation is proportional to its size.
- as a model for **radioactive decay** with $\lambda < 0$: the rate of decay is proportional to the number of atoms.

LINEAR DIFFERENTIAL EQUATIONS IN HIGHER DIMENSIONS. Linear dynamical systems have the form $\dot{x} = Ax$, where A is a matrix. Note that the origin $\vec{0}$ is an **equilibrium point**: if $x(0) = 0$, then $x(t) = 0$ for all t . The general solution is $x(t) = e^{At} = 1 + At + A^2t^2/2! + \dots$ because $\dot{x}(t) = A + 2A^2t/2! + \dots = A(1 + At + A^2t^2/2! + \dots) = Ae^{At} = Ax(t)$. If $B = S^{-1}AS$ is diagonal with the eigenvalues $\lambda_j = a_j + ib_j$ in the diagonal, then $y = S^{-1}x$ satisfies $y(t) = e^{Bt}$ and therefore $y_j(t) = e^{\lambda_j t} y_j(0) = e^{a_j t} e^{ib_j t} y_j(0)$. The solutions in the original coordinates are $x(t) = Sy(t)$.

PHASE PORTRAITS. For differential equations $\dot{x} = f(x)$ in 2D one can **draw the vector field** $x \mapsto f(x)$. The solution $x(t)$ is tangent to the vector $f(x(t))$ everywhere. The phase portraits together with some solution curves reveal much about the system. Some examples of phase portraits of linear two dimensional systems



UNDERSTANDING A DIFFERENTIAL EQUATION. The closed form solution like $x(t) = e^{At}x(0)$ for $\dot{x} = Ax$ is actually quite useless. One wants to understand the solution quantitatively. Questions one wants to answer are: what happens in the long term? Is the origin stable, are there periodic solutions. Can one decompose the system into simpler subsystems? We will see that **diagonalisation** allows to **understand the system**: by decomposing it into one-dimensional linear systems, which can be analyzed seperately. In general, "understanding" can mean different things:

- Plotting phase portraits.
- Computing solutions numerically and estimate the error.
- Finding special solutions.
- Predicting the shape of some orbits.
- Finding regions which are invariant.

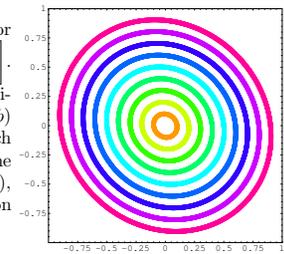
- Finding special closed form solutions $x(t)$.
- Finding a power series $x(t) = \sum_n a_n t^n$ in t .
- Finding quantities which are unchanged along the flow (called "Integrals").
- Finding quantities which increase along the flow (called "Lyapunov functions").

LINEAR STABILITY. A linear dynamical system $\dot{x} = Ax$ with diagonalizable A is linearly stable if and only if $a_j = \text{Re}(\lambda_j) < 0$ for all eigenvalues λ_j of A .

PROOF. We see that from the explicit solutions $y_j(t) = e^{a_j t} e^{ib_j t} y_j(0)$ in the basis consisting of eigenvectors. Now, $y(t) \rightarrow 0$ if and only if $a_j < 0$ for all j and $x(t) = Sy(t) \rightarrow 0$ if and only if $y(t) \rightarrow 0$.

RELATION WITH DISCRETE TIME SYSTEMS. From $\dot{x} = Ax$, we obtain $x(t+1) = Bx(t)$, with the matrix $B = e^A$. The eigenvalues of B are $\mu_j = e^{\lambda_j}$. Now $|\mu_j| < 1$ if and only if $\text{Re} \lambda_j < 0$. The criterium for linear stability of discrete dynamical systems is compatible with the criterium for linear stability of $\dot{x} = Ax$.

EXAMPLE 1. The system $\dot{x} = y, \dot{y} = -x$ can in vector form $v = (x, y)$ be written as $\dot{v} = Av$, with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The matrix A has the eigenvalues $i, -i$. After a coordinate transformation $w = S^{-1}v$ we get with $w = (a, b)$ the differential equations $\dot{a} = ia, \dot{b} = -ib = b$ which has the solutions $a(t) = e^{it}a(0), b(t) = e^{-it}b(0)$. The original coordinates satisfy $x(t) = \cos(t)x(0) - \sin(t)y(0), y(t) = \sin(t)x(0) + \cos(t)y(0)$. Indeed e^{At} is a rotation in the plane.



EXAMPLE 2. A **harmonic oscillator** $\ddot{x} = -x$ can be written with $y = \dot{x}$ as $\dot{x} = y, \dot{y} = -x$ (see Example 1). The general solution is therefore $x(t) = \cos(t)x(0) - \sin(t)\dot{x}(0)$.

EXAMPLE 3. We take **two harmonic oscillators and couple them**: $\ddot{x}_1 = -x_1 - \epsilon(x_2 - x_1), \ddot{x}_2 = -x_2 + \epsilon(x_2 - x_1)$. For small x_i one can simulate this with two coupled penduli. The system can be written as $\ddot{v} = Av$, with $A = \begin{bmatrix} -1 + \epsilon & -\epsilon \\ -\epsilon & -1 + \epsilon \end{bmatrix}$. The matrix A has an eigenvalue $\lambda_1 = -1$ to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and an eigenvalue $\lambda_2 = -1 + 2 * \epsilon$ to the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The coordinate change S is $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. It has the inverse $S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} / 2$. In the coordinates $w = S^{-1}v = [y_1, y_2]$, we have oscillations $\ddot{y}_1 = -y_1$ corresponding to the case $x_1 - x_2 = 0$ (the pendula swing synchronous) and $\ddot{y}_2 = -(1 - 2\epsilon)y_2$ corresponding to $x_1 + x_2 = 0$ (the pendula swing against each other).

COMPLEX LINEAR 1D CASE. $\dot{x} = \lambda x$ for $\lambda = a + ib$ has solution $x(t) = e^{at}e^{ibt}x(0)$ and norm $\|x(t)\| = e^{at}\|x(0)\|$.

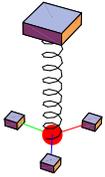
OSCILLATOR: The system $\ddot{x} = -\lambda x$ has the solution $x(t) = \cos(\sqrt{\lambda}t)x(0) + \sin(\sqrt{\lambda}t)\dot{x}(0)/\sqrt{\lambda}$.

DERIVATION. $\dot{x} = y, \dot{y} = -\lambda x$ and in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

and because A has eigenvalues $\pm i\sqrt{\lambda}$, the new coordinates move as $a(t) = e^{i\sqrt{\lambda}t}a(0)$ and $b(t) = e^{-i\sqrt{\lambda}t}b(0)$.

Writing this in the original coordinates $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$ and fixing the constants gives $x(t), y(t)$.



EXAMPLE. THE SPINNER. The spinner is a rigid body attached to a spring aligned around the z-axis. The body can rotate around the z-axis and bounce up and down. The two motions are coupled in the following way: when the spinner winds up in the same direction as the spring, the spring gets tightend and the body gets a lift. If the spinner winds up to the other direction, the spring becomes more relaxed and the body is lowered. Instead of reducing the system to a 4D first order system, system $\frac{d}{dt}\vec{x} = A\vec{x}$, we will keep the second time derivative and diagonalize the 2D system $\frac{d^2}{dt^2}\vec{x} = A\vec{x}$, where we know how to solve the one dimensional case $\frac{d^2}{dt^2}v = -\lambda v$ as $v(t) = A \cos(\sqrt{\lambda}t) + B \sin(\sqrt{\lambda}t)$ with constants A, B depending on the initial conditions, $v(0), \dot{v}(0)$.

SETTING UP THE DIFFERENTIAL EQUATION.

x is the angle and y the height of the body. We put the coordinate system so that $y = 0$ is the point, where the body stays at rest if $x = 0$. We assume that if the spring is winded up with an angle x , this produces an upwards force x and a momentum force $-3x$. We furthermore assume that if the body is at position y , then this produces a momentum y onto the body and an upwards force y . The differential equations

$$\begin{aligned} \ddot{x} &= -3x + y \\ \ddot{y} &= -y + x \end{aligned} \quad \text{can be written as } \ddot{v} = Av = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} v.$$

FINDING GOOD COORDINATES $w = S^{-1}v$ is obtained with getting the eigenvalues and eigenvectors of A : $\lambda_1 = -2 - \sqrt{2}, \lambda_2 = -2 + \sqrt{2}$ $v_1 = \begin{bmatrix} -1 - \sqrt{2} \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 + \sqrt{2} \\ 1 \end{bmatrix}$ so that $S = \begin{bmatrix} -1 - \sqrt{2} & -1 + \sqrt{2} \\ 1 & 1 \end{bmatrix}$.

SOLVE THE SYSTEM $\ddot{a} = \lambda_1 a, \ddot{b} = \lambda_2 b$ IN THE GOOD COORDINATES $\begin{bmatrix} a \\ b \end{bmatrix} = S^{-1} \begin{bmatrix} x \\ y \end{bmatrix}$.
 $a(t) = A \cos(\omega_1 t) + B \sin(\omega_1 t), \omega_1 = \sqrt{-\lambda_1}, b(t) = C \cos(\omega_2 t) + D \sin(\omega_2 t), \omega_2 = \sqrt{-\lambda_2}$.

THE SOLUTION IN THE ORIGINAL COORDINATES. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$. At $t = 0$ we know $x(0), y(0), \dot{x}(0), \dot{y}(0)$. This fixes the constants
 in $x(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)$. The curve $(x(t), y(t))$ traces a Lyssajoux curve:

ASYMPTOTIC STABILITY.

A linear system $\dot{x} = Ax$ in the 2D plane is asymptotically stable if and only if $\det(A) > 0$ and $\text{tr}(A) < 0$.

PROOF. If the eigenvalues λ_1, λ_2 of A are real then both being negative is equivalent with $\lambda_1 \lambda_2 = \det(A) > 0$ and $\text{tr}(A) = \lambda_1 + \lambda_2 < 0$. If $\lambda_1 = a + ib, \lambda_2 = a - ib$, then a negative a is equivalent to $\lambda_1 + \lambda_2 = 2a < 0$ and $\lambda_1 \lambda_2 = a^2 + b^2 > 0$.

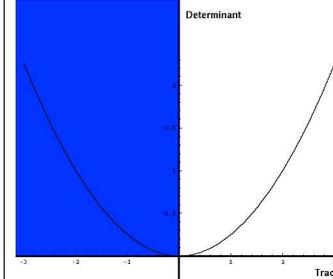
ASYMPTOTIC STABILITY COMPARISON OF DISCRETE AND CONTINUOUS SITUATION.

The trace and the determinant are independent of the basis, they can be computed fast, and are real if A is real. It is therefore convenient to determine the region in the $\text{tr} - \det$ -plane, where continuous or discrete dynamical systems are asymptotically stable. While the continuous dynamical system is related to a discrete system, it is important not to mix these two situations up.

Continuous dynamical system.

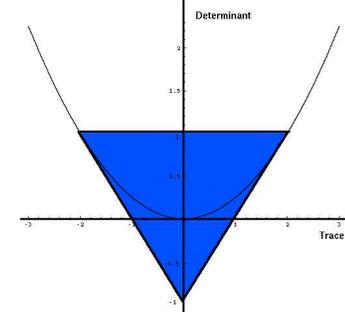
Discrete dynamical system.

Stability of $\dot{x} = Ax$ ($x(t+1) = e^A x(t)$).



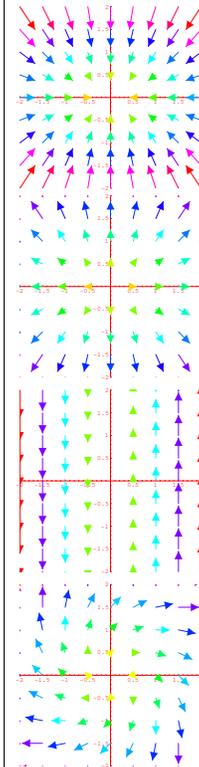
Stability in $\det(A) > 0, \text{tr}(A) < 0$
 Stability if $\text{Re}(\lambda_1) < 0, \text{Re}(\lambda_2) < 0$.

Stability of $x(t+1) = Ax$



Stability in $|\det(A)| < 1 < \det(A) < 1$
 Stability if $|\lambda_1| < 1, |\lambda_2| < 1$.

PHASEPORTRAITS. (In two dimensions we can plot the vector field, draw some trajectories)

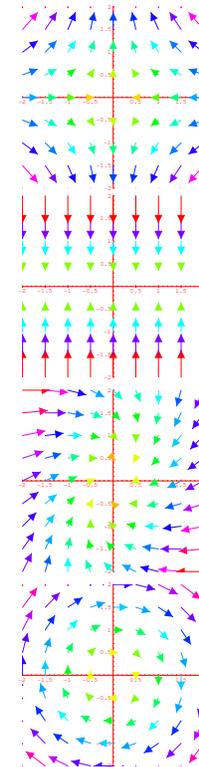


$\lambda_1 < 0, \lambda_2 < 0$,
 i.e. $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

$\lambda_1 > 0, \lambda_2 > 0$,
 i.e. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\lambda_1 = 0, \lambda_2 = 0$,
 i.e. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\lambda_1 = a + ib, a > 0, \lambda_2 = a - ib$,
 i.e. $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$



$\lambda_1 < 0, \lambda_2 > 0$,
 i.e. $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

$\lambda_1 = 0, \lambda_2 < 0$,
 i.e. $A = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}$

$\lambda_1 = a + ib, a < 0, \lambda_2 = a - ib$,
 i.e. $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$\lambda_1 = ib, \lambda_2 = -ib$,
 i.e. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

SUMMARY. For linear systems $\dot{x} = Ax$, the eigenvalues of A determine the behavior completely. For nonlinear systems explicit formulas for solutions are no more available in general. It even also happen that orbits go off to infinity in finite time like in $\dot{x} = x^2$ with solution $x(t) = -1/(t - x(0))$. With $x(0) = 1$ it reaches infinity at time $t = 1$. Linearity is often too crude. The exponential growth $\dot{x} = ax$ of a bacteria colony for example is slowed down due to lack of food and the **logistic model** $\dot{x} = ax(1 - x/M)$ would be more accurate, where M is the population size for which bacteria starve so much that the growth has stopped: $x(t) = M$, then $\dot{x}(t) = 0$. Nonlinear systems can be investigated with **qualitative methods**. In 2 dimensions $\dot{x} = f(x, y), \dot{y} = g(x, y)$, where chaos does not happen, the analysis of **equilibrium points** and **linear approximation** at those points in general allows to understand the system. Also in higher dimensions, where ODE's can have chaotic solutions, the analysis of equilibrium points and linear approximation at those points is a place, where linear algebra becomes useful.

EQUILIBRIUM POINTS. A point x_0 is called an **equilibrium point** of $\dot{x} = f(x)$ if $f(x_0) = 0$. If $x(0) = x_0$ then $x(t) = x_0$ for all times. The system $\dot{x} = x(6 - 2x - y), \dot{y} = y(4 - x - y)$ for example has the four equilibrium points $(0, 0), (3, 0), (0, 4), (2, 2)$.

JACOBIAN MATRIX. If x_0 is an equilibrium point for $\dot{x} = f(x)$ then $[A]_{ij} = \frac{\partial f_i}{\partial x_j}(x_0)$ is called the **Jacobian** at x_0 . For two dimensional systems

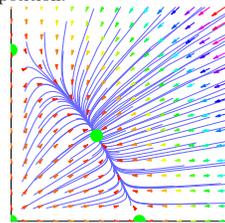
$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \quad \text{this is the } 2 \times 2 \text{ matrix} \quad A = \begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\ \frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y) \end{bmatrix}.$$

The linear ODE $\dot{y} = Ay$ with $y = x - x_0$ approximates the nonlinear system well near the equilibrium point. The Jacobian is the linear approximation of $F = (f, g)$ near x_0 .

VECTOR FIELD. In two dimensions, we can draw the vector field by hand: attaching a vector $(f(x, y), g(x, y))$ at each point (x, y) . To find the equilibrium points, it helps to draw the **nullclines** $\{f(x, y) = 0\}, \{g(x, y) = 0\}$. The equilibrium points are located on intersections of nullclines. The eigenvalues of the Jacobians at equilibrium points allow to draw the vector field near equilibrium points. This information is sometimes enough to draw the vector field **by hand**.

MURRAY SYSTEM (see handout) $\dot{x} = x(6 - 2x - y), \dot{y} = y(4 - x - y)$ has the nullclines $x = 0, y = 0, 2x + y = 6, x + y = 5$. There are 4 equilibrium points $(0, 0), (3, 0), (0, 4), (2, 2)$. The Jacobian matrix of the system at the point (x_0, y_0) is $\begin{bmatrix} 6 - 4x_0 - y_0 & -x_0 \\ -y_0 & 4 - x_0 - 2y_0 \end{bmatrix}$. Note that without interaction, the two systems would be logistic systems $\dot{x} = x(6 - 2x), \dot{y} = y(4 - y)$. The additional $-xy$ is the competition.

Equilibrium	Jacobian	Eigenvalues	Nature of equilibrium
$(0, 0)$	$\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$	$\lambda_1 = 6, \lambda_2 = 4$	Unstable source
$(3, 0)$	$\begin{bmatrix} -6 & -3 \\ 0 & 1 \end{bmatrix}$	$\lambda_1 = -6, \lambda_2 = 1$	Hyperbolic saddle
$(0, 4)$	$\begin{bmatrix} 2 & 0 \\ -4 & -4 \end{bmatrix}$	$\lambda_1 = 2, \lambda_2 = -4$	Hyperbolic saddle
$(2, 2)$	$\begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$	$\lambda_i = -3 \pm \sqrt{5}$	Stable sink



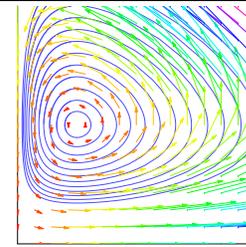
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USING TECHNOLOGY (Example: Mathematica). Plot the vector field:
Needs["Graphics`PlotField`"]
f[x_, y_] := {x(6-2x-y), y(5-x-y)}; PlotVectorField[f[x, y], {x, 0, 4}, {y, 0, 4}]
Find the equilibrium solutions:
Solve[{x(6-2x-y)==0, y(5-x-y)==0}, {x, y}]
Find the Jacobian and its eigenvalues at (2, 2):
A[{x_, y_}] := {{6-4x, -x}, {-y, 5-x-2y}}; Eigenvalues[A[{2, 2}]]
Plotting an orbit:
S[u_, v_] := NDSolve[{x'[t]==x[t](6-2x[t]-y[t]), y'[t]==y[t](5-x[t]-y[t]), x[0]==u, y[0]==v}, {x, y}, {t, 0, 1}]
ParametricPlot[Evaluate[{x[t], y[t]}/.S[0.3, 0.5]], {t, 0, 1}, AspectRatio->1, AxesLabel->{"x[t]", "y[t]"}]
    
```

VOLTERRA-LODKA SYSTEMS are systems of the form

$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy \end{aligned}$$

This example has equilibrium points $(0, 0)$ and $(1/2, 1)$.



It describes for example a tuna shark population. The tuna population $x(t)$ becomes smaller with more sharks. The shark population grows with more tuna. Volterra explained so first the oscillation of fish populations in the Mediterranean sea.

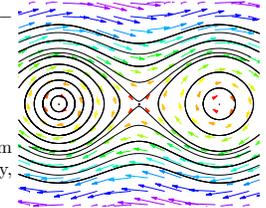
EXAMPLE: HAMILTONIAN SYSTEMS are systems of the form

$$\begin{aligned} \dot{x} &= \partial_y H(x, y) \\ \dot{y} &= -\partial_x H(x, y) \end{aligned} \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x) \end{aligned}$$

where H is called the **energy**. Usually, x is the position and y the momentum.

THE PENDULUM: $H(x, y) = y^2/2 - \cos(x)$.

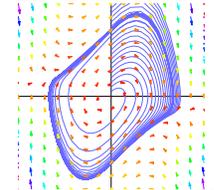
x is the angle between the pendulum and the x -axis, y is the angular velocity, $\sin(x)$ is the potential.



(See homework). Hamiltonian systems preserve energy $H(x, y)$ because $\frac{d}{dt}H(x(t), y(t)) = \partial_x H(x, y)\dot{x} + \partial_y H(x, y)\dot{y} = \partial_x H(x, y)\partial_y H(x, y) - \partial_y H(x, y)\partial_x H(x, y) = 0$. Orbits stay on level curves of H .

EXAMPLE: LIENHARD SYSTEMS are differential equations of the form $\ddot{x} + \dot{x}F'(x) + G'(x) = 0$. With $y = \dot{x} + F(x), G'(x) = g(x)$, this gives

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -g(x) \end{aligned} \quad \begin{aligned} \dot{x} &= y - (x^3/3 - x) \\ \dot{y} &= -x \end{aligned}$$



Lienhard systems have **limit cycles**. A trajectory always ends up on that limit cycle. This is useful for engineers, who need oscillators which are stable under changes of parameters. One knows: if $g(x) > 0$ for $x > 0$ and F has exactly three zeros $0, a, -a, F'(0) < 0$ and $F'(x) \geq 0$ for $x > a$ and $F(x) \rightarrow \infty$ for $x \rightarrow \infty$, then the corresponding Lienhard system has exactly one stable limit cycle.

CHAOS can occur for systems $\dot{x} = f(x)$ in three dimensions. For example, $\ddot{x} = f(x, t)$ can be written with $(x, y, z) = (x, \dot{x}, t)$ as $(\dot{x}, \dot{y}, \dot{z}) = (y, f(x, z), 1)$. The system $\ddot{x} = f(x, \dot{x})$ becomes in the coordinates (x, \dot{x}) the ODE $\dot{x} = f(x)$ in four dimensions. The term **chaos** has no uniform definition, but usually means that one can find a copy of a random number generator embedded inside the system. Chaos theory is more than 100 years old. Basic insight had been obtained by Poincaré. During the last 30 years, the subject exploded to its own branch of physics, partly due to the availability of computers.

ROESSLER SYSTEM

$$\begin{aligned} \dot{x} &= -(y + z) \\ \dot{y} &= x + y/5 \\ \dot{z} &= 1/5 + xz - 5.7z \end{aligned}$$



LORENTZ SYSTEM

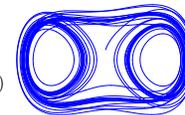
$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= -xz + 28x - y \\ \dot{z} &= xy - \frac{8z}{3} \end{aligned}$$



These two systems are examples, where one can observe **strange attractors**.

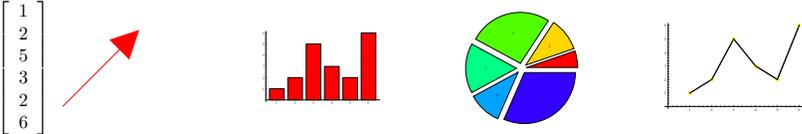
THE DUFFING SYSTEM

$$\begin{aligned} \ddot{x} + \frac{\dot{x}}{10} - x + x^3 - 12 \cos(t) &= 0 \\ \dot{x} &= y \\ \dot{y} &= -y/10 - x + x^3 - 12 \cos(t) \\ \dot{z} &= 1 \end{aligned}$$



The Duffing system models a metallic plate between magnets. Other chaotic examples can be obtained from mechanics like the **driven pendulum** $\ddot{x} + \sin(x) - \cos(t) = 0$.

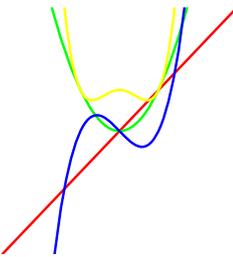
FROM VECTORS TO FUNCTIONS. Vectors can be displayed in different ways:



Listing the (i, \vec{v}_i) can also be interpreted as the graph of a function $f : 1, 2, 3, 4, 5, 6 \rightarrow \mathbf{R}$, where $f(i) = \vec{v}_i$.

LINEAR SPACES. A space X in which we can add, scalar multiplications and where basic laws like commutativity, distributivity and associativity hold is called a **linear space**. Examples:

- Lines, planes and more generally, the n -dimensional Euclidean space.
- P_n , the space of all polynomials of degree n .
- The space P of all polynomials.
- C^∞ , the space of all smooth functions on the line
- C^0 , the space of all continuous functions on the line.
- C^1 , the space of all differentiable functions on the line.
- $C^\infty(\mathbf{R}^3)$ the space of all smooth functions in space.
- L^2 the space of all functions on the line for which f^2 is integrable and $\int_{-\infty}^{\infty} f^2(x) dx < \infty$.



In all these function spaces, the function $f(x) = 0$ which is constantly 0 is the zero function.

WHICH OF THE FOLLOWING ARE LINEAR SPACES?

- The space X of all polynomials of the form $f(x) = ax^3 + bx^4 + cx^5$
- The space X of all continuous functions on the unit interval $[-1, 1]$ which vanish at -1 and 1 . It contains for example $f(x) = x^2 - |x|$.
- The space X of all smooth periodic functions $f(x+1) = f(x)$. Example $f(x) = \sin(2\pi x) + \cos(6\pi x)$.
- The space $X = \sin(x) + C^\infty(\mathbf{R})$ of all smooth functions $f(x) = \sin(x) + g$, where g is a smooth function.
- The space X of all trigonometric polynomials $f(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + \dots + a_n \sin(nx)$.
- The space X of all smooth functions on \mathbf{R} which satisfy $f(1) = 1$. It contains for example $f(x) = 1 + \sin(x) + x$.
- The space X of all continuous functions on \mathbf{R} which satisfy $f(2) = 0$ and $f(10) = 0$.
- The space X of all smooth functions on \mathbf{R} which satisfy $\lim_{|x| \rightarrow \infty} f(x) = 0$.
- The space X of all continuous functions on \mathbf{R} which satisfy $\lim_{|x| \rightarrow \infty} f(x) = 1$.
- The space X of all smooth functions on \mathbf{R} of compact support: for every f , there exists an interval I such that $f(x) = 0$ outside that interval.
- The space X of all smooth functions on \mathbf{R}^2 .

LINEAR TRANSFORMATIONS. A map T between linear spaces is called **linear** if $T(x + y) = T(x) + T(y), T(\lambda x) = \lambda T(x), T(0) = 0$. Examples:

- $Df(x) = f'(x)$ on C^∞
- $Tf(x) = \int_0^x f(x) dx$ on C^0
- $Tf(x) = (f(0), f(1), f(2), f(3))$ on C^∞ .
- $Tf(x) = \sin(x)f(x)$ on C^∞
- $Tf(x) = (\int_0^1 f(x)g(x) dx)g(x)$ on $C^0[0, 1]$.

WHICH OF THE FOLLOWING MAPS ARE LINEAR TRANSFORMATIONS?

- The map $T(f) = f'(x)$ on $X = C^\infty(\mathbf{T})$.
- The map $T(f) = 1 + f'(x)$ on $X = C^\infty(\mathbf{R})$.
- The map $T(f)(x) = \sin(x)f(x)$ on $X = C^\infty(\mathbf{R})$.
- The map $T(f)(x) = f(x)/x$ on $X = C^\infty(\mathbf{R})$.
- The map $T(f)(x) = \int_0^x f(x) dx$ on $X = C([0, 1])$.
- The map $T(f)(x) = f(x + \sqrt{2})$ on $X = C^\infty(\mathbf{R})$.
- The map $T(f)(x) = \int_{-\infty}^{\infty} f(x-s) \sin(s) ds$ on C^0 .
- The map $T(f)(x) = f'(x) + f(2)$ on $C^\infty(\mathbf{R})$.
- The map $T(f)(x) = f''(x) + f(2) + 1$ on $C^\infty(\mathbf{R})$.
- The map $T(f)(x, y, z) = f_{xx}(x, y, z) + f_{yy}(x, y, z) + f_{zz}(x, y, z) + 1/(|x|)f(x, y, z)$
- The map $T(f)(x) = f(x^2)$.
- The map $T(f)(x) = f''(x) - x^2 f(x)$.
- The map $T(f)(x) = f^2(x)$ on $C^\infty(\mathbf{R})$.

EIGENVALUES, BASIS, KERNEL, IMAGE. Many concepts work also here.

X linear space	$f, g \in X, f + g \in X, \lambda f \in X, 0 \in X$.
T linear transformation	$T(f + g) = T(f) + T(g), T(\lambda f) = \lambda T(f), T(0) = 0$.
f_1, f_2, \dots, f_n linear independent	$\sum_i c_i f_i = 0$ implies $f_i = 0$.
f_1, f_2, \dots, f_n span X	Every f is of the form $\sum_i c_i f_i$.
f_1, f_2, \dots, f_n basis of X	linear independent and span.
T has eigenvalue λ	$Tf = \lambda f$
kernel of T	$\{Tf = 0\}$
image of T	$\{Tf f \in X\}$.

Some concepts do not work without modification. Example: $\det(T)$ or $\text{tr}(T)$ are not always defined for linear transformations in infinite dimensions. The concept of a basis in infinite dimensions has to be defined properly.

INNER PRODUCT. The analogue of the dot product $\sum_i f_i g_i$ for vectors $(f_1, f_2, \dots, f_n), (g_1, g_2, \dots, g_n)$ is the integral $\int_0^1 f(x)g(x) dx$ for functions on the interval $[0, 1]$. One writes (f, g) and $\|f\| = \sqrt{(f, f)}$. This inner product is defined on L^2 or $C^\infty([0, 1])$. Example: $(x^3, x^2) = \int_0^1 x^5 dx = 1/5$. Having an inner product allows to define "length", "angle" in infinite dimensions. One can use it to define projections, Gram-Schmidt orthogonalization etc.

DIFFERENTIAL EQUATIONS, 12/5/2003

Math 21b, O. Knill

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS. $Df = Tf = f'$ is a linear map on the space of smooth functions C^∞ . If $p(x) = a_0 + a_1x + \dots + a_nx^n$ is a polynomial, then $p(D) = a_0 + a_1D + \dots + a_nD^n$ is a linear map on $C^\infty(\mathbf{R})$ too. We will see here how to find the general solution of $p(D)f = g$.

EXAMPLE. For $p(x) = x^2 - x + 6$ and $g(x) = \cos(x)$ the problem $p(D)f = g$ is the differential equation $f''(x) - f'(x) - 6f(x) = \cos(x)$. It has the solution $c_1e^{-2x} + c_2e^{3x} - (\sin(x) + 7\cos(x))/50$, where c_1, c_2 are arbitrary constants. How do we find these solutions?

THE IDEA. In general, a differential equation $p(D)f = g$ has many solution. For example, for $p(D) = D^3$, the equation $D^3f = 0$ has solutions $(c_0 + c_1x + c_2x^2)$. The constants come from integrating three times. Integrating means applying D^{-1} but since D has as a kernel the constant functions, integration gives a one dimensional space of anti-derivatives (we can add a constant to the result and still have an anti-derivative). In order to solve $D^3f = g$, we integrate g three times. We will generalize this idea by writing $T = p(D)$ as a product of simpler transformations which we can invert. These simpler transformations have the form $(D - \lambda)f = g$.

FINDING THE KERNEL OF A POLYNOMIAL IN D . How do we find a basis for the kernel of $Tf = f'' + 2f' + f$? The linear map T can be written as a polynomial in D which means $T = D^2 - D - 2 = (D + 1)(D - 2)$. The kernel of T contains the kernel of $D - 2$ which is one-dimensional and spanned by $f_1 = e^{2x}$. The kernel of $T = (D - 2)(D + 1)$ also contains the kernel of $D + 1$ which is spanned by $f_2 = e^{-x}$. The kernel of T is therefore two dimensional and spanned by e^{2x} and e^{-x} .

THEOREM: If $T = p(D) = D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0$ on C^∞ then $\dim(\ker(T)) = n$.

PROOF. $T = p(D) = \prod(D - \lambda_j)$, where λ_j are the roots of the polynomial p . The kernel of T contains the kernel of $D - \lambda_j$ which is spanned by $f_j(t) = e^{\lambda_j t}$. In the case when we have a factor $(D - \lambda_j)^k$ of T , then we have to consider the kernel of $(D - \lambda_j)^k$ which is $q(t)e^{\lambda_j t}$, where q is a polynomial of degree $k - 1$. For example, the kernel of $(D - 1)^3$ consists of all functions $(a + bt + ct^2)e^t$.

SECOND PROOF. Write this as $A\dot{g} = 0$, where A is a $n \times n$ matrix and $g = [f, \dot{f}, \dots, f^{(n-1)}]^T$, where $f^{(k)} = D^k f$ is the k 'th derivative. The linear map $T = AD$ acts on vectors of functions. If all eigenvalues λ_j of A are different (they are the same λ_j as before), then A can be diagonalized. Solving the diagonal case $BD = 0$ is easy. It has a n dimensional kernel of vectors $F = [f_1, \dots, f_n]^T$, where $f_i(t) = t$. If $B = SAS^{-1}$, and F is in the kernel of BD , then SF is in the kernel of AD .

REMARK. The result can be generalized to the case, when a_j are functions of x . Especially, $Tf = g$ has a solution, when T is of the above form. It is important that the function in front of the highest power D^n is bounded away from 0 for all t . For example $x Df(x) = e^x$ has no solution in C^∞ , because we can not integrate e^x/x . An example of a ODE with variable coefficients is the **Sturm-Liouville** eigenvalue problem $T(f)(x) = a(x)f''(x) + a'(x)f'(x) + q(x)f(x) = \lambda f(x)$ like for example the Legendre differential equation $(1 - x^2)f''(x) - 2xf'(x) + n(n + 1)f(x) = 0$.

BACKUP

- Equations $Tf = 0$, where $T = p(D)$ form **linear differential equations with constant coefficients** for which we want to understand the solution space. Such equations are called **homogeneous**. **Solving differential equations often means finding a basis of the kernel of T** . In the above example, a general solution of $f'' + 2f' + f = 0$ can be written as $f(t) = a_1f_1(t) + a_2f_2(t)$. If we fix two values like $f(0), f'(0)$ or $f(0), f(1)$, the solution is unique.
- If we want to solve $Tf = g$, an **inhomogeneous equation** then T^{-1} is not unique because we have a kernel. If g is in the image of T there is at least one solution f . The general solution is then $f + \ker(T)$. For example, for $T = D^2$, which has C^∞ as its image, we can find a solution to $D^2f = t^3$ by integrating twice: $f(t) = t^5/20$. The kernel of T consists of all linear functions $at + b$. The general solution to $D^2 = t^3$ is $at + b + t^5/20$. The integration constants parameterize actually the kernel of a linear map.

THE SYSTEM $Tf = (D - \lambda)f = g$ has the general solution $\int_0^x (ce^{\lambda x} + e^{\lambda x} \int_0^x e^{-\lambda t} g(t) dt)$. The solution $f = (D - \lambda)^{-1}g$ is the sum of a function in the kernel and a special function.

THE SOLUTION OF $(D - \lambda)^k f = g$ is obtained by applying $(D - \lambda)^{-1}$ several times on g . In particular, for $g = 0$, we get: the kernel of $(D - \lambda)^k$ as $(c_0 + c_1x + \dots + c_{k-1}x^{k-1})e^{\lambda x}$.

THEOREM. The inhomogeneous $p(D)f = g$ has an n -dimensional space of solutions in $C^\infty(\mathbf{R})$.

PROOF. To solve $Tf = p(D)f = g$, we write the equation as $(D - \lambda_1)^{k_1}(D - \lambda_2)^{k_2} \dots (D - \lambda_n)^{k_n} f = g$. Since we know how to invert each $T_j = (D - \lambda_j)^{k_j}$, we can construct the general solution by inverting one factor T_j of T one after another. Often we can find directly a special solution f_1 of $p(D)f = g$ and get the general solution as $f_1 + f_h$, where f_h is in the n -dimensional kernel of T .

EXAMPLE 1) $Tf = e^{3x}$, where $T = D^2 - D = D(D - 1)$. We first solve $(D - 1)f = e^{3x}$. It has the solution $f_1 = ce^x + e^x \int_0^x e^{-t} e^{3t} dt = c_2e^x + e^{3x}/2$. Now solve $Df = f_1$. It has the solution $c_1 + c_2e^x + e^{3x}/6$.

EXAMPLE 2) $Tf = \sin(x)$ with $T = (D^2 - 2D + 1) = (D - 1)^2$. We see that $\cos(x)/2$ is a special solution. The kernel of $T = (D - 1)^2$ is spanned by xe^x and e^x so that the general solution is $(c_1 + c_2x)e^x + \cos(x)/2$.

EXAMPLE 3) $Tf = x$ with $T = D^2 + 1 = (D - i)(D + i)$ has the special solution $f(x) = x$. The kernel is spanned by e^{ix} and e^{-ix} or also by $\cos(x), \sin(x)$. The general solution can be written as $c_1 \cos(x) + c_2 \sin(x) + x$.

EXAMPLE 4) $Tf = x$ with $T = D^4 + 2D^2 + 1 = (D - i)^2(D + i)^2$ has the special solution $f(x) = x$. The kernel is spanned by $e^{ix}, xe^{ix}, e^{-ix}, x^{-ix}$ or also by $\cos(x), \sin(x), x \cos(x), x \sin(x)$. The general solution can be written as $(c_0 + c_1x) \cos(x) + (d_0 + d_1x) \sin(x) + x$.

THESE EXAMPLES FORM 4 TYPICAL CASES.

CASE 1) $p(D) = (D - \lambda_1) \dots (D - \lambda_n)$ with real λ_i . The general solution of $p(D)f = g$ is the sum of a special solution and $c_1e^{\lambda_1x} + \dots + c_n e^{\lambda_nx}$

CASE 2) $p(D) = (D - \lambda)^k$. The general solution is the sum of a special solution and a term $(c_0 + c_1x + \dots + c_{k-1}x^{k-1})e^{\lambda x}$

CASE 3) $p(D) = (D - \lambda)(D - \bar{\lambda})$ with $\lambda = a + ib$. The general solution is a sum of a special solution and a term $c_1e^{ax} \cos(bx) + c_2e^{ax} \sin(bx)$

CASE 4) $p(D) = (D - \lambda)^k(D - \bar{\lambda})^k$ with $\lambda = a + ib$. The general solution is a sum of a special solution and $(c_0 + c_1x + \dots + c_{k-1}x^{k-1})e^{ax} \cos(bx) + (d_0 + d_1x + \dots + d_{k-1}x^{k-1})e^{ax} \sin(bx)$

We know this also from the eigenvalue problem for a matrix. We either have distinct real eigenvalues, or we have some eigenvalues with multiplicity, or we have pairs of complex conjugate eigenvalues which are distinct, or we have pairs of complex conjugate eigenvalues with some multiplicity.

CAS SOLUTION OF ODE's: Example: DSolve[f''[x] - f'[x] == Exp[3x], f[x], x]

REMARK. (informal) Operator methods are also useful for ODEs with variable coefficients. For example, $T = H - 1 = D^2 - x^2 - 1$, the **quantum harmonic oscillator**, can be written as $T = A^*A = AA^* + 2$ with a **creation operator** $A^* = (D - x)$ and **annihilation operator** $A = (D + x)$. (Hint: use the **commutation relation** $Dx - xD = 1$.) The kernel $f_0 = Ce^{-x^2/2}$ of $A = (D + x)$ is also the kernel of T and so an eigenvector of T and H . It is called the **vacuum**. If f is an eigenvector of H with $Hf = \lambda f$, then A^*f is an eigenvector with eigenvalue $\lambda + 2$. Proof. Because $HA^* - A^*H = [H, A^*] = 2A^*$, we have $H(A^*f) = A^*Hf + [H, A^*]f = A^*\lambda f + 2A^*f = (\lambda + 2)(A^*f)$. We obtain all eigenvectors $f_n = A^*f_{n-1}$ of eigenvalue $\lambda + 2n$ by applying iteratively the creation operator A^* on the vacuum f_0 . Because every function f with $\int f^2 dx < \infty$ can be written uniquely as $f = \sum_{n=0}^\infty a_n f_n$, we can **diagonalize** H and solve $Hf = g$ with $f = \sum_n b_n / (1 + 2n) f_n$, where $g = \sum_n b_n f_n$.

HOW DOES IT WORK? Piecewise smooth functions $f(x)$ on $[-\pi, \pi]$ form a linear space. The numbers $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx} dx$ are called the **Fourier coefficients** of f . We can recover f from these coefficients by $f(x) = \sum_k c_k e^{ikx}$. When we write $f = f_{\text{even}} + f_{\text{odd}}$, where $f_{\text{even}}(x) = (f(x) + f(-x))/2$ and $f_{\text{odd}}(x) = (f(x) - f(-x))/2$, then $f_{\text{even}}(x) = a_0/\sqrt{2} + \sum_{k=1}^{\infty} a_k \cos(kx)$ and $f_{\text{odd}}(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$. The sum of the cos-series for f_{even} and the sin-series for f_{odd} up gives the **real Fourier series** of f . Those coefficients are $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)/\sqrt{2} dx$, $b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$.

While the complex expansion is more elegant, the real expansion has the advantage that one does not leave the real numbers.

SUMMARY.

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx} dx.$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)/\sqrt{2} dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$$

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

EXAMPLE 1. Let $f(x) = x$ on $[-\pi, \pi]$. This is an odd function ($f(-x) + f(x) = 0$) so that it has a sin series: with $b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(kx) dx = \frac{1}{\pi} (x \cos(kx)/k + \sin(kx)/k^2) \Big|_{-\pi}^{\pi} = 2(-1)^{k+1}/k$ we get $x = \sum_{k=1}^{\infty} 2 \frac{(-1)^{k+1}}{k} \sin(kx)$. For example, $\pi/2 = 2(1 - 1/3 + 1/5 - 1/7 + \dots)$ a **formula of Leibnitz**.

EXAMPLE 2. Let $f(x) = \cos(x) + 1/7 \cos(5x)$. This **trigonometric polynomial** is already the Fourier series. The nonzero coefficients are $a_1 = 1, a_5 = 1/7$.

EXAMPLE 3. Let $f(x) = |x|$ on $[-\pi, \pi]$. This is an even function ($f(-x) - f(x) = 0$) so that it has a cos series: with $a_0 = 1/(2\sqrt{2}), a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(kx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(kx) dx = \frac{2}{\pi} (x \sin(kx)/k + \cos(kx)/k^2) \Big|_0^{\pi} = \frac{2}{\pi} ((-1)^k - 1)/k^2$ for $k > 0$.

WHERE ARE FOURIER SERIES USEFUL? Examples:

- **Partial differential equations.** PDE's like $\ddot{u} = c^2 u''$ become ODE's: $\ddot{u}_k = c^2 k^2 u_k$ can be solved as $u_k(t) = a_k \sin(ckt) u_k(0)$ and lead to solutions $u(t, x) = \sum_k a_k \sin(ckt) \sin(kx)$ of the PDE.
- **Sound** Coefficients a_k form the **frequency spectrum** of a sound f . **Filters** suppress frequencies, **equalizers** transform the Fourier space, **compressors** (i.e.MP3) select frequencies relevant to the ear.
- **Analysis:** $\sum_k a_k \sin(kx) = f(x)$ give explicit expressions for sums which would be hard to evaluate otherwise. The Leibnitz sum $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \dots$ (Example 1 above) is an example.
- **Number theory:** Example: if α is irrational, then the fact that $n\alpha \pmod{1}$ are uniformly distributed in $[0, 1]$ can be understood with Fourier theory.
- **Chaos theory:** Quite many notions in Chaos theory can be defined or analyzed using Fourier theory. Examples are mixing properties or ergodicity.
- **Quantum dynamics:** Transport properties of materials are related to spectral questions for their Hamiltonians. The relation is given by Fourier theory.
- **Crystallography:** X ray Diffraction patterns of a crystal, analyzed using Fourier theory reveal the structure of the crystal.
- **Probability theory:** The Fourier transform $\chi_X = E[e^{iX}]$ of a random variable is called **characteristic function**. Independent case: $\chi_{x+y} = \chi_x \chi_y$.
- **Image formats:** like JPG compress by cutting irrelevant parts in Fourier space.

WHY DOES IT WORK? One has a **dot product** on functions with $f \cdot g = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)}g(x) dx$. The functions $e_k = e^{ikx}$ serve as **basis vectors**. As in Euclidean space, where $v \cdot e_k = v_k$ is the k 'th **coordinate** of v , the Fourier coefficient $c_k = e_k \cdot f = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx$ is the k -th coordinate of f . In the same way as we wrote $v = \sum_k v_k e_k$ in Euclidean space, we have $f(x) = \sum_k c_k e^{ikx}(x)$. The vectors e_k are orthonormal: $\frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{e_k(x)} e_m(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(m-k)ix} dx$ which is zero for $m \neq k$ and 1 for $m = k$.

Also the functions $1/\sqrt{2}, \sin(nx), \cos(nx)$ form an **orthonormal basis** with respect to the dot product $(f, g) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$. The Fourier coefficients a_0, a_n, b_n are the coordinates of f in that basis.

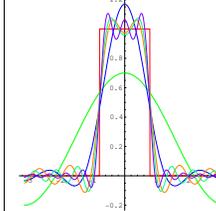
Fourier basis is not the only one. An other basis which has many applications are **Wavelet expansions**. The principle is the same.

LENGTH, DISTANCE AND ANGLE. With a dot product, one has a **length** $\|f\|^2 = f \cdot f = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$ and can measure **distances** between two functions as $\|f - g\|$ and **angles** between functions f, g by $\cos(\alpha) = f \cdot g / (\|f\| \|g\|)$. Functions are called **orthogonal** if $(f, g) = f \cdot g = 0$. For example, an even function f and an odd function g are orthogonal. The functions e^{inx} form an orthonormal family. The vectors $\sqrt{2} \cos(kx), \sqrt{2} \sin(kx)$ form an orthonormal family, $\cos(kx)$ is in the linear space of **even functions** and $\sin(kx)$ in the linear space of **odd functions**. If we work with sin or cos Fourier series, we use the dot product $f \cdot g = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx$ for which the functions $1/\sqrt{2}, \cos(kx), \sin(kx)$ are **orthonormal**.

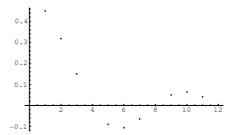
REWRITING THE DOT PRODUCT. If $f(x) = \sum_k c_k e^{ikx}$ and $g(x) = \sum_k d_k e^{ikx}$, then $\frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x)}g(x) dx = \frac{1}{2\pi} \int \sum_{k,m} \overline{c_k} e^{-ikx} d_m e^{imx} = \sum_k \overline{c_k} d_k$. The dot product is the sum of the product of the coordinates as in finite dimensions. If f and g are even, then $f = f_0/\sqrt{2} + \sum_{k=1}^{\infty} f_k \cos(kx), g = g_0/\sqrt{2} + \sum_{k=1}^{\infty} g_k \cos(kx)$ and $\frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx = \sum_{k=0}^{\infty} f_k g_k$. If f and g are odd, and $f = \sum_k f_k \sin(kx), g = \sum_{k=1}^{\infty} g_k \sin(kx)$ then $(f, g) = f \cdot g = \sum_{k=1}^{\infty} f_k g_k$. Especially the Parseval equality $\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=1}^{\infty} f_k^2$ holds. It can be useful also to find closed formulas for the sum of some series.

EXAMPLE. $f(x) = x = 2(\sin(x) - \sin(2x))/2 + \sin(3x)/3 - \sin(4x)/4 + \dots$ has coefficients $f_k = 2(-1)^{k+1}/k$ and so $4(1 + 1/4 + 1/9 + \dots) = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = 2\pi^2/3$ or $1 + 1/4 + 1/9 + 1/16 + 1/25 + \dots = \pi^2/6$.

APPROXIMATIONS.



If $f(x) = \sum_k b_k \cos(kx)$, then $f_n(x) = \sum_{k=1}^n b_k \cos(kx)$ is an approximation to f . Because $\|f - f_n\|^2 = \sum_{k=n+1}^{\infty} b_k^2$ goes to zero, the graphs of the functions f_n come for large n close to the graph of the function f . The picture to the left shows an approximation of a piecewise continuous even function, the right hand side the values of the coefficients $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$.



SOME HISTORY. The **Greeks** approximation of planetary motion through **epicycles** was an early use of Fourier theory: $z(t) = e^{it}$ is a circle (Aristarchus system), $z(t) = e^{it} + e^{imt}$ is an epicycle (Ptolemaeus system), **18'th century** Mathematicians like Euler, Lagrange, Bernoulli knew experimentally that Fourier series worked.



Fourier's (picture left) claim of the convergence of the series was confirmed in the **19'th century** by Cauchy and Dirichlet. For continuous functions the sum does not need to converge everywhere. However, as the 19 year old Fejér (picture right) demonstrated in his theses in 1900, the coefficients still determine the function $\sum_{k=-n}^{n-1} \frac{n-|k|}{n} f_k e^{ikx} \rightarrow f(x)$ for $n \rightarrow \infty$ if f is continuous and $f(-\pi) = f(\pi)$. Partial differential equations, i.e. the **theory of heat** had sparked early research in Fourier theory.



OTHER FOURIER TRANSFORMS. On a finite interval one obtains a series, on the line an integral, on finite sets, finite sums. The **discrete Fourier transformation** (DFT) is important for applications. It can be determined efficiently by the (FFT=**F**ast **F**ourier **T**ransform) found in 1965, reducing the n^2 steps to $n \log(n)$.

Domain	Name	Synthesis	Coefficients
$T = [-\pi, \pi]$	Fourier series	$f(x) = \sum_k \hat{f}_k e^{ikx}$	$\hat{f}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$
$\mathbf{R} = (-\infty, \infty)$	Fourier transforms	$f(x) = \int_{-\infty}^{\infty} f(k) e^{ikx} dx$	$f(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$
$\mathbf{Z}_n = \{1, \dots, n\}$	DFT	$f_m = \sum_{k=1}^n \hat{f}_k e^{imk2\pi/n}$	$\hat{f}_k = \frac{1}{n} \sum_{m=1}^n f_m e^{-ikm2\pi/n}$

All these transformations can be defined in dimension d . Then $k = (k_1, \dots, k_d)$ etc. are vectors. 2D DFT is for example useful in **image manipulation**.

COMPUTER ALGEBRA. Packages like Mathematica have the discrete Fourier transform built in `Fourier[{0.3, 0.4, 0.5}]` for example, gives the DFT of a three vector. You can perform a simple Fourier analysis yourself by listening to a sound like `Play[Sin[2000 * x * Floor[7 * x]/12], {x, 0, 20}]` ...

HEAT AND WAVE EQUATION 12/10/2003

Math 21b, O. Knill

NOTATION. We write $T_x(x, t)$ and $T_t(x, t)$ for the partial derivatives with respect to x or t .

THE HEAT EQUATION. The temperature distribution $T(x, t)$ in a metal bar $[0, T]$ satisfies the **heat equation** $T_t(x, t) = \mu T_{xx}(x, t)$. At every point, the rate of change of the temperature is proportional to the second derivative of $x \dots T(x, t)$ at x . The function $T(t, x)$ is zero at both ends of the bar and $f(x) = T(0, x)$ is a given initial temperature distribution.

SOLVING IT. This **partial differential equation** (PDE) is solved by writing $T(x, t) = u(x)v(t)$ which leads to $\dot{v}(t)u(x) = v(t)u''(x)$ or $\dot{v}(t)/(\mu v(t)) = u''(x)/u(x)$. Because the LHS does not depend on x and the RHS not on t , this must be a constant λ . The ODE $u''(x) = \lambda u(x)$ is an eigenvalue problem $D^2 u = \lambda u$ which has only solutions when $\lambda = -k^2$ for integers k . The eigenfunctions of D^2 are $u_k(x) = \sin(kx)$. The second equation $\dot{v}(t) = k^2 \mu v$ is solved by $v(t) = e^{-k^2 \mu t}$ so that $u(x, t) = \sin(kx)e^{-k^2 t}$ is a solution of the heat equation. Because linear combinations of solutions are solutions too, a general solution is of the form $T(x, t) = \sum_{k=1}^{\infty} a_k \sin(kx)e^{-k^2 t}$. For fixed t this is a **Fourier series**. The coefficients a_k are determined from $f(x) = T(x, 0) = \sum_{k=1}^{\infty} a_k \sin(kx)$, where $a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)$. (The factor 2 comes from the fact that f should be thought to be extended to an odd function on $[-\pi, \pi]$ and the integral from $[-\pi, 0]$ is the same as the from $[0, \pi]$.)

FOURIER: (Solving the heat equation was the birth of Fourier theory) The heat equation $T_t(t, x) = \mu T_{xx}(t, x)$ with smooth $T(x, 0) = f(x), T(0, 0) = T(\pi, 0) = 0$ has the solution $\sum_{k=1}^{\infty} a_k \sin(kx)e^{-k^2 \mu t}$ with $a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) dx$.

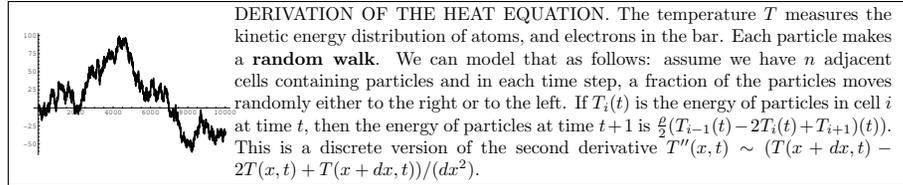
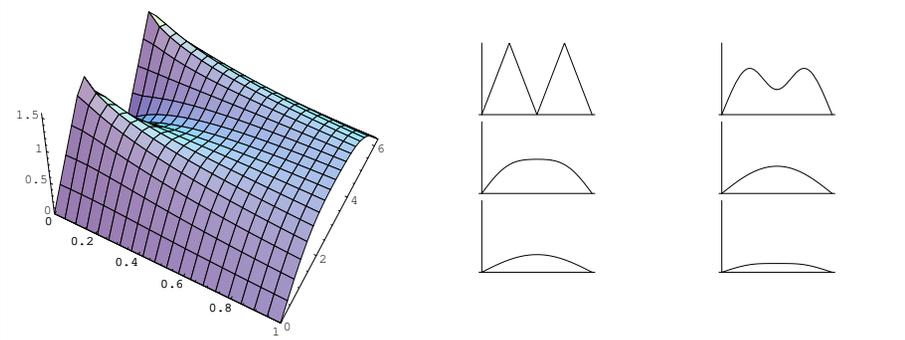
Remark. If $T(0, 0) = a, T(\pi, 0) = b$ instead, then the solution is $T(x, t) = T_0(x, t) + a + bx/\pi$, where $T_0(x, t)$ is the solution in the box using $f_0(x) = f(x) - (a + bx/\pi)$ which satisfies $f_0(0) = f_0(\pi) = 0$. (Because $T(t, x) = a + bx/\pi$ is a solution of the heat equation, we can add it to any solution and get new solutions. This allows to tune the boundary conditions).

EXAMPLE. $f(x) = x$ on $[-\pi/2, \pi/2]$ π -periodically continued has the Fourier coefficients $a_k = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} x \sin(kx) dx = 4/(k^2 \pi)(-1)^{(k-1)/2}$ for odd k and 0 else. The solution is

$$T(x, t) = \sum_{m=0}^{\infty} \frac{4(-1)^m}{(2m+1)^2 \pi} e^{-\mu(2m+1)^2 t} \sin((2m+1)x).$$

The exponential term containing the time makes the function $T(x, t)$ converge to 0: "The bar cools." The higher frequency terms are damped faster because "Smaller disturbances are smoothed out faster."

VISUALIZATION. We can just plot the graph of the function $T(x, t)$ or plot the temperature distribution for different times t_i .



FOURIER TRANSFORM=DIAGONALISATION OF D. Fourier theory actually **diagonalizes** the linear map D : write $f = \sum_n \hat{f}_n e^{inx} = S^{-1}f$ and $Sf = \hat{f} = (\dots, c_{-1}, c_0, c_1, \dots)$. $SDS^{-1}\hat{f} = SD\sum_n \hat{f}_n e^{inx} = S\sum_n n\hat{f}_n e^{inx} = (\dots, i(-1)c_{-1}, 0c_0, ic_1, \dots)$. This means that differentiation changes the Fourier coefficients as $c_n \mapsto inc_n$. From this we see that the diagonalisation of D^2 is multiplying with $-n^2$. Now, if $\dot{u} = D^2 u$, then $\dot{\hat{u}}_n = -\mu n^2 \hat{u}_n$ so that $\hat{u}_n(t) = e^{-\mu n^2 t} \hat{u}_n(0)$. The partial differential equation is transformed into a sequence of ordinary differential equations.

Differentiation of the function	$\begin{matrix} f \\ D \downarrow \\ Df \end{matrix}$	$\begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix}$	$\begin{matrix} \hat{f} \\ \downarrow M \\ M\hat{f} \end{matrix}$	becomes multiplication with the diagonal matrix	$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2i & 0 & 0 & 0 & 0 \\ \cdot & 0 & i & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 & i & 0 \\ \cdot & 0 & 0 & 0 & 0 & 2i \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$
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WAVE EQUATION. The wave equation $T_{tt}(x, t) = c^2 T_{xx}(x, t)$ is solved in a similar way as the heat equation: writing $T(x, t) = u(x)v(t)$ gives $\ddot{v}u = c^2 v u''$ or $\ddot{v}/(c^2 v) = u''/u = -k^2 = const$. The right equation has solutions $u_k(x) = \sin(kx)$. Now, $v_k(t) = a_k \cos(kct) + b_k \sin(kct)$ solves $\ddot{v} = -c^2 k^2 v$ so that $T_k(x, t) = u_k(x)v_k(t) = \sin(kx)(a_k \cos(kct) + b_k \sin(kct))$ is a solution of the wave equation. General solutions can be obtained by taking **superpositions of these waves** $T(x, t) = \sum_{k=1}^{\infty} a_k \sin(kx) \cos(kct) + b_k \sin(kx) \sin(kct)$. The coefficients a_k, b_k are obtained from $T(x, 0) = \sum_{k=1}^{\infty} a_k \sin(kx)$ and $\dot{T}(x, 0) = \sum_{k=1}^{\infty} kcb_k \sin(kx)$. Unlike in the heat equation both $T(x, 0)$ and $\dot{T}(x, 0)$ have to be specified.

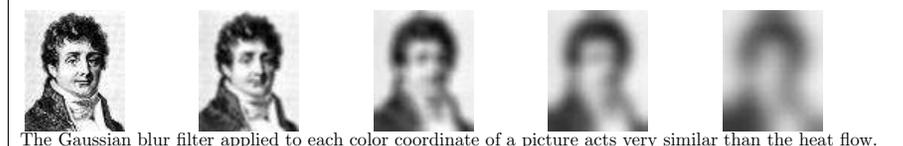
WAVE EQUATION: The wave equation $T_{tt} = c^2 T_{xx}$ with smooth $T(x, 0) = f(x), \dot{T}(x, 0) = g(x), T(0, 0) = T(\pi, 0) = 0$ has the solution $\sum_{k=1}^{\infty} a_k \sin(kx) \cos(kct) + b_k \sin(kx) \sin(kct)$ with $a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx) dx$ and $kcb_k = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(kx) dx$.

QUANTUM MECHANICS = MATRIX MECHANICS A particle in a box is described by a linear map $L = -D^2 \hbar^2 / (2m)$ acting on functions on $[0, \pi]$. In the Fourier picture, this linear map is

$M = \frac{\hbar^2}{2m} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -4 & 0 & 0 & 0 & 0 \\ \cdot & 0 & -1 & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 & 0 & 0 \\ \cdot & 0 & 0 & 0 & -1 & 0 \\ \cdot & 0 & 0 & 0 & 0 & -4 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$	The diagonal entries are the eigenvalues of L as usual after diagonalisation.
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Quantum mech. observables are symmetric linear maps: for example: position, $X : f \mapsto xf$, momentum $P : f \mapsto i\hbar \nabla f$, kinetic energy $f \mapsto P^2/2m = -\hbar^2 \Delta / (2m)$, Coulomb potential energy $f \mapsto (4\pi\epsilon_0)^{-1} e^2 / r f$.

HIGHER DIMENSIONS. The heat or wave equations can be solved also in **higher dimensions** using Fourier theory. For example, on a metallic plate $[-\pi, \pi] \times [-\pi, \pi]$ the temperature distribution $T(x, y, t)$ satisfies $T_t = \mu(T_{xx} + T_{yy})$. The Fourier series of a function $f(x, y)$ in two variables is $\sum_{n, m} c_{n, m} e^{inx+my}$, where $c_{n, m} = (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x, y) e^{-inx} e^{-imy} dx dy$.



The Gaussian blur filter applied to each color coordinate of a picture acts very similar to the heat flow.

LAPLACE EQUATION (informal) 12/12/2003

Math 21b, O. Knill

LAPLACE EQUATION. The linear map $T \mapsto \Delta T = T_{xx} + T_{yy}$ for smooth functions in the plane is called the **Laplacian**. The PDE $\Delta T = 0$ is called the **Laplace equation**. A solution T is in the kernel of Δ and called **harmonic**. For example $T(x, y) = x^2 - y^2$ is harmonic. One can show that $T(x, y) = \text{Re}((x + iy)^n)$ or $T(x, y) = \text{Im}((x + iy)^n)$ are all harmonic.

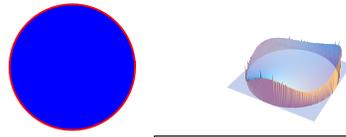
DIRICHLET PROBLEM: find a function $T(x, y)$ on a region Ω which satisfies $\Delta T(x, y) = 0$ inside Ω and which is prescribed function $f(x, y)$ at the boundary. This **boundary value problem** can be solved explicitly in many cases.

IN A DISC. $\{x^2 + y^2 \leq 1\}$. If $f(t) = T(\cos(t), \sin(t))$ is prescribed on the boundary and T satisfies $\Delta T = 0$ inside the disc, then T can be found via Fourier theory:

If $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$, then $T(x, y) = a_0 + \sum_{n=1}^{\infty} a_n \text{Re}((x + iy)^n) + b_n \text{Im}((x + iy)^n)$ satisfies the Laplace equation and $T(\cos(t), \sin(t)) = f(t)$ on the boundary of the disc.

PROOF. If $f = a_0$ is constant on the boundary then $T(x, y) = a_0$. If $f(t) = \sin(nt)$ on the boundary $z = e^{it} = x + iy$ then $T(x, y) = \text{Im}(z^n)$ and if $f(t) = \cos(nt)$ on the boundary then $T(x, y) = \text{Re}(z^n)$ is a solution.

EXAMPLE. $\text{Re}(z^2) = x^2 - y^2 + i(2xy)$ gives the two harmonic functions $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$: they satisfy $(\partial_x^2 + \partial_y^2)f = 0$.



ESPECIALLY: the mean value property for harmonic functions $T(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt$ holds.

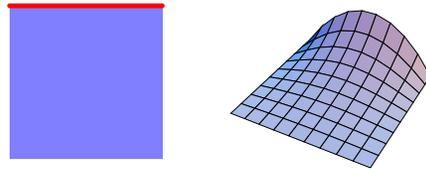
POISSON FORMULA: With $z = x + iy = re^{i\theta}$ and $T(x, y) = f(z)$, the general solution is given also by the **Poisson integral formula**:

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{r^2 - 2r \cos(\theta - \phi) + 1} f(e^{i\phi}) d\phi.$$

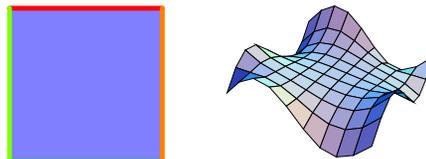
PHYSICAL RELEVANCE. If the **temperature** or **charge** distribution on the boundary of a region Ω , then $T(x, y)$ is the stable temperature or **charge** distribution inside that region. In the later case, the electric field inside Ω is then ∇U . An other application is that the velocity v of an ideal incompressible fluid satisfies $v = \nabla U$, where U satisfies the Laplace differential equation $\Delta U = 0$ in the region.

SQUARE, SPECIAL CASE. We solve first the case, when T vanishes on three sides $x = 0, x = \pi, y = 0$ and $T(x, \pi) = f(x)$. Separation of variables gives then $T(x, y) = \sum_n a_n \sin(nx) \sinh(ny)$, where the coefficients a_n are obtained from $T(x, \pi) = \sum_n b_n \sin(nx) = \sum_n a_n \sin(nx) \sinh(n\pi)$, where $b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx$ are the Fourier coefficients of f . The solution is therefore

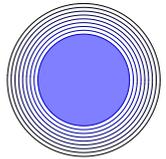
$$T_f(x, y) = \sum_{n=1}^{\infty} b_n \sin(nx) \sinh(ny) / \sinh(n\pi)$$



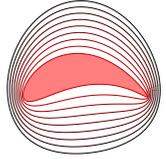
SQUARE GENERAL CASE Solutions to general boundary conditions can be obtained by adding the solutions in the following cases: $T_f(x, \pi - y)$ solves the case, when T vanishes on the sides $x = 0, x = \pi, y = \pi$ and is f on $y = 0$. The function $T_f(y, x)$ solves the case, when T vanishes on the sides $y = 0, y = \pi, x = 0$ and $T_f(\pi - y, x)$ solves the case, when T vanishes on $y = 0, y = \pi$ and $x = \pi$. The general solution is $T(x, y) = T_f(x, y) + T_g(x, \pi - y) + T_h(y, x) + T_k(\pi - y, x)$



A GENERAL REGION. For a general region, one uses numerical methods. One possibility is by **conformal transportation**: to map the region into the the disc using a complex map F , which maps the region Ω into the disc. Solve then the problem in the disc with boundary value $T(F^{-1}(x, y))$ on the disc.



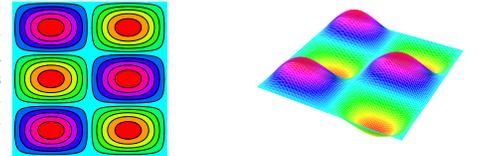
If $S(x, y)$ is the solution there, then $T(x, y) = S(F^{-1}(x, y))$ is the solution in the region Ω . The picture shows the example of the Joukowski map $z \mapsto (z + 1/z)/2$ which has been used in fluid dynamics (N.J. Joukowski (1847-1921) was a Russian aerodynamics researcher.)



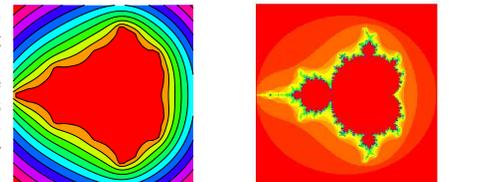
A tourists view on related PDE topics

POISSON EQUATION. The Poisson equation $\Delta f = g$ in a square $[0, \pi]^2$ can be solved via Fourier theory: If $f(x, y) = \sum_{k,m} a_{n,m} \cos(nx) \cos(my)$ and $g(x, y) = \sum_{k,m} b_{n,m} \cos(nx) \cos(my)$, then $f_{xx} + f_{yy} = \sum_{n,m} -(n^2 + m^2) a_{n,m} \cos(nx) \cos(my) = \sum_{k,m} b_{n,m} \cos(nx) \cos(my)$, then $a_{n,m} = -b_{n,m} / (n^2 + m^2)$. The poisson equation is important in **electrostatics**, for example to determine the electromagnetic field when the charge and current distribution is known: $\Delta U(x) = -(1/\epsilon_0)\rho$, then $E = \nabla U$ is the electric field to the charge distribution $\rho(x)$.

EIGENVALUES. The functions $\sin(nx) \sin(ny)$ are eigenfunctions for the Laplace operator $f_{xx} + f_{yy} = n^2 f$ on the disc. For a general bounded region Ω , we can look at all smooth functions which are zero on the boundary of Ω . The possible eigenvalues of $\Delta f = \lambda f$ are the possible energies of a particle in Ω .

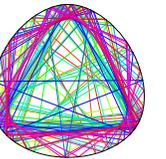


COMPLEX DYNAMICS. Also in complex dynamics, harmonic functions appear. Finding properties of complicated sets like the **Mandelbrot set** is done by mapping the exterior to the outside of the unit circle. If the Mandelbrot set is charged then the contour lines of equal potential can be obtained as the corresponding contour lines in the disc case (where the lines are circles).

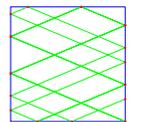


SCHRÖDINGER EQUATION. If H is the energy operator, then $i\hbar \dot{f} = Hf$ is called the **Schrödinger equation**. If $H = -\hbar^2 \Delta / (2m)$, this looks very much like the heat equation, if there were not the i . If f is an eigenvalue of H , then $i\hbar \dot{f} = \lambda f$ and $f(t) = e^{i\lambda t} f(0)$. In the heat equation, we would get $f(t) = e^{-\mu(t)} f(0)$. The evolution of the Schrödinger equation is very similar to the wave equation.

QUANTUM CHAOS studies the eigenvalues and eigenvectors of the Laplacian in a bounded region. If the billiard in the region is chaotic, the study of the eigensystem is called **quantum chaos**. The eigenvalue problem $\Delta f = \lambda f$ and the **billiard problem** in the region are related. A famous open problem is whether two smooth convex regions in the plane for which the eigenvalues λ_j are the same are equivalent up to rotation and translation.



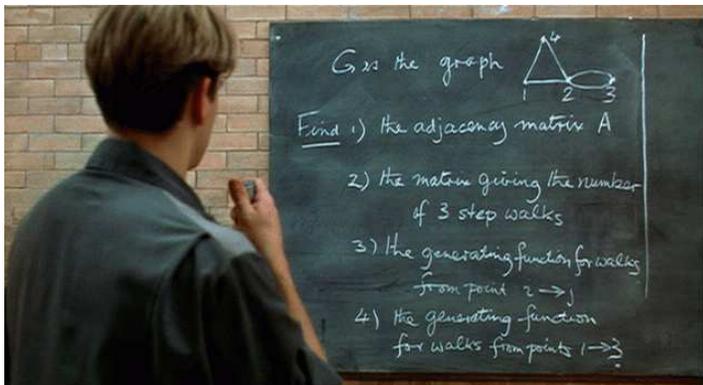
In a square of size L we know all the eigenvalues $\frac{\pi^2}{L^2} (n^2 + m^2)$. The eigenvectors are $\sin(\frac{p\pi}{L} nx) \sin(\frac{q\pi}{L} mx)$. We have explicit formulas for the eigenfunctions and eigenvalues. On the classical level, for the billiard, we have also a complete picture, how a billiard path will look like. The square is an example, where we have no quantum chaos.



ICE: A blackboard problem

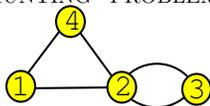
Math 21b, O. Knill, Fall 2003

In the movie "Good Will Hunting", the main character Will Hunting (Matt Damon) solves a blackboard challenge problem, which is given as a challenge to a linear algebra class.



THE "WILL HUNTING" PROBLEM.

G is the graph



Find.

- 1) the adjacency matrix A .
- 2) the matrix giving the number of 3 step walks.
- 3) the generating function for walks from point $i \rightarrow j$.
- 4) the generating function for walks from points $1 \rightarrow 3$.

This problem belongs to linear algebra and calculus evenso the problem origins from graph theory or combinatorics. For a calculus student who has never seen the connection between graph theory, calculus and linear algebra, the assignment is actually hard - probably too hard - as the movie correctly indicates. The problem was posed in the last part of a linear algebra course.

An explanation of some terms:

THE ADJACENCY MATRIX. The structure of the graph can be encoded with a 4×4 array which encodes how many paths of length 1, one can take in the graph from one node to another:

no	one	no	one
one	none	two	one
no	two	no	no
one	one	no	no

which can more conveniently be written as an array of numbers called a **matrix**:

$$L = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Problem 2 asks to find the matrix which encodes all possible paths of length 3.

GENERATING FUNCTION. To any graph one can assign for every pair of nodes i, j a series $f(z) = \sum_{n=0}^{\infty} a_n^{(ij)} z^n$, where $a_n^{(ij)}$ is the number of possible walks from node i to node j with n steps. Problem 3) asks for an explicit expression of $f(z)$ and problem 4) asks for an explicit expression in the special case $i = 1, j = 3$.

Linear algebra has many relations to other fields in mathematics. It is not true that linear algebra is just about solving systems of linear equations.

SOLUTION TO 2). $[L^2]_{ij}$ is by definition of the matrix product the sum $L_{i1}L_{1j} + L_{i2}L_{2j} + \dots + L_{i4}L_{4j}$. Each term $L_{i1}L_{1j}$ is 1 if and only if there is a path of length 2 going from i to j passing through k . Therefore $[L^2]_{ij}$ is the number of paths of length 2 going from node i to j . Similarly, $[L^n]_{ij}$ is the number of paths of length n going from i to j . The answer is

$$L^3 = \begin{bmatrix} 2 & 7 & 2 & 3 \\ 7 & 2 & 12 & 7 \\ 2 & 12 & 0 & 2 \\ 3 & 7 & 2 & 2 \end{bmatrix}$$

SOLUTION TO 3). The geometric series formula

$$\sum_n x^n = (1 - x)^{-1}$$

holds also for matrices:

$$f(z) = \sum_{n=0}^{\infty} [L^n]_{ij} z^n = \left[\sum_{n=0}^{\infty} L^n z^n \right]_{ij} = (1 - Lz)^{-1}$$

Cramers formula for the inverse of a matrix which involves the determinant

$$A^{-1} = \det(A_{ij}) / \det(A)$$

This leads to an explicit formula

$$\det(1 - zL_{ij}) / \det(1 - zL)$$

which can also be written as

$$\det(L_{ij} - z) / \det(L - z)$$

SOLUTION TO 4). This is the special case, when $i = 1$ and $j = 3$.

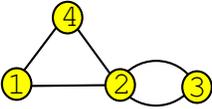
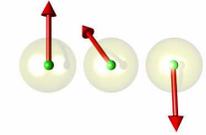
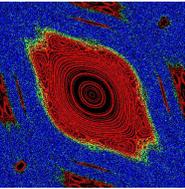
$$\det(L - z) = \det \begin{pmatrix} 1 & -z & 1 \\ 0 & 2 & 0 \\ 1 & 1 & -z \end{pmatrix} = -2 - z + 3z^2 - z^3$$

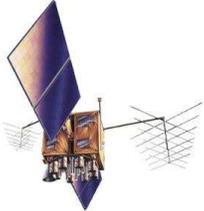
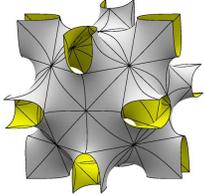
$$\det(L - z) = \det \begin{pmatrix} -z & 1 & 0 & 1 \\ 1 & -z & 2 & 1 \\ 0 & 2 & -z & 0 \\ 1 & 1 & 0 & -z \end{pmatrix} = 4 - 2z - 7z^2 + z^4$$

The final result is

$$(z^3 + 3z^2 - z - 2) / (z^4 - 7z^2 - 2z + 4)$$

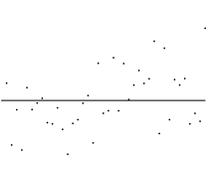
This is **not** a list of topics covered in the course. It is rather a loose selection of subjects, in which linear algebra is useful or relevant. The aim is to convince you that it is worth learning this subject. Most likely, some of this handout does not make much sense yet to you. Look at this page at the end of the course again, some of the content will become more interesting then.

	<p>GRAPHS, NETWORKS. Linear algebra can be used to understand networks which is a collection of nodes connected by edges. Networks are also called graphs. The adjacency matrix of a graph is defined by $A_{ij} = 1$ if there is an edge from node i to node j in the graph. Otherwise the entry is zero. A problem using such matrices appeared on a blackboard at MIT in the movie "Good will hunting".</p>	<p>Application: A_{ij}^n is the number of n-step walks in the graph which start at the vertex i and end at the vertex j.</p>
	<p>CHEMISTRY, MECHANICS Complicated objects like a bridge (the picture shows Storrow Drive connection bridge which is part of the "big dig"), or a molecule (i.e. a protein) can be modeled by finitely many parts (bridge elements or atoms) coupled with attractive and repelling forces. The vibrations of the system are described by a differential equation $\dot{x} = Ax$, where $x(t)$ is a vector which depends on time. Differential equations are an important part of this course.</p>	<p>The solution $x(t) = \exp(At)$ of the differential equation $\dot{x} = Ax$ can be understood and computed by finding the eigenvalues of the matrix A. Knowing these frequencies is important for the design of a mechanical object because the engineer can damp dangerous frequencies. In chemistry or medicine, the knowledge of the vibration resonances allows to determine the shape of a molecule.</p>
	<p>QUANTUM COMPUTING A quantum computer is a quantum mechanical system which is used to perform computations. The state x of a machine is no more a sequence of bits like in a classical computer but a sequence of qubits, where each qubit is a vector. The memory of the computer can be represented as a vector. Each computation step is a multiplication $x \mapsto Ax$ with a suitable matrix A.</p>	<p>Theoretically, quantum computations could speed up conventional computations significantly. They could be used for example for cryptological purposes. Freely available quantum computer language (QCL) interpreters can simulate quantum computers with an arbitrary number of qubits.</p>
	<p>CHAOS THEORY. Dynamical systems theory deals with the iteration of maps or the analysis of solutions of differential equations. At each time t, one has a map $T(t)$ on the vector space. The linear approximation $DT(t)$ is called Jacobean is a matrix. If the largest eigenvalue of $DT(t)$ grows exponentially in t, then the system shows "sensitive dependence on initial conditions" which is also called "chaos".</p>	<p>Examples of dynamical systems are our solar system or the stars in a galaxy, electrons in a plasma or particles in a fluid. The theoretical study is intrinsically linked to linear algebra because stability properties often depends on linear approximations.</p>

	<p>CODING, ERROR CORRECTION Coding theory is used for encryption or error correction. In the first case, the data x are mapped by a map T into code $y=Tx$. For a good code, T is a "trapdoor function" in the sense that it is hard to get x back when y is known. In the second case, a code is a linear subspace X of a vector space and T is a map describing the transmission with errors. The projection of Tx onto the subspace X corrects the error.</p>	<p>Linear algebra enters in different ways, often directly because the objects are vectors but also indirectly like for example in algorithms which aim at cracking encryption schemes.</p>
	<p>DATA COMPRESSION Image- (i.e. JPG), video- (MPG4) and sound compression algorithms (i.e. MP3) make use of linear transformations like the Fourier transform. In all cases, the compression makes use of the fact that in the Fourier space, information can be cut away without disturbing the main information.</p>	<p>Typically, a picture, a sound or a movie is cut into smaller junks. These parts are represented by vectors. If U denotes the Fourier transform and P is a cutoff function, then $y = PUx$ is transferred or stored on a CD or DVD. The receiver obtains back $U^T y$ which is close to x in the sense that the human eye or ear does not notice a big difference.</p>
	<p>SOLVING SYSTEMS OR EQUATIONS When extremizing a function f on data which satisfy a constraint $g(x) = 0$, the method of Lagrange multipliers asks to solve a nonlinear system of equations $\nabla f(x) = \lambda \nabla g(x), g(x) = 0$ for the $(n + 1)$ unknowns (x, λ), where ∇f is the gradient of f.</p>	<p>Solving systems of nonlinear equations can be tricky. Already for systems of polynomial equations, one has to work with linear spaces of polynomials. Even if the Lagrange system is a linear system, the task of solving it can be done more efficiently using a solid foundation of linear algebra.</p>
	<p>GAMES Moving around in a world described in a computer game requires rotations and translations to be implemented efficiently. Hardware acceleration can help to handle this.</p>	<p>Rotations are represented by orthogonal matrices. For example, if an object located at $(0, 0, 0)$, turning around the y-axes by an angle ϕ, every point in the object gets transformed by the matrix</p> $\begin{pmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{pmatrix}$

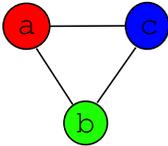
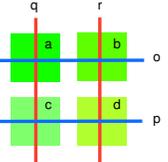
USE OF LINEAR ALGEBRA (III)

Math 21b, Oliver Knill

	<p>STATISTICS When analyzing data statistically, one often is interested in the correlation matrix $A_{ij} = E[Y_i Y_j]$ of a random vector $X = (X_1, \dots, X_n)$ with $Y_i = X_i - E[X_i]$. This matrix is derived from the data and determines often the random variables when the type of the distribution is fixed.</p>	<p>For example, if the random variables have a Gaussian (=Bell shaped) distribution, the correlation matrix together with the expectation $E[X_i]$ determines the random variables.</p>
	<p>GAME THEORY Abstract Games are often represented by pay-off matrices. These matrices tell the outcome when the decisions of each player are known.</p>	<p>A famous example is the prisoner dilemma. Each player has the choice to cooperate or to cheat.. The game is described by a 2x2 matrix like for example $\begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix}$. If a player cooperates and his partner also, both get 3 points. If his partner cheats and he cooperates, he gets 5 points. If both cheat, both get 1 point. More generally, in a game with two players where each player can chose from n strategies, the pay-off matrix is a n times n matrix A. A Nash equilibrium is a vector $p \in S = \{\sum_i p_i = 1, p_i \geq 0\}$ for which $qAp \leq pAp$ for all $q \in S$.</p>
	<p>NEURAL NETWORK In part of neural network theory, for example Hopfield networks, the state space is a 2n-dimensional vector space. Every state of the network is given by a vector x, where each component takes the values -1 or 1. If W is a symmetric nxn matrix, one can define a "learning map" $T : x \mapsto \text{sign}Wx$, where the sign is taken component wise. The energy of the state is the dot product $-(x, Wx)/2$. One is interested in fixed points of the map.</p>	<p>For example, if $W_{ij} = x_i y_j$, then x is a fixed point of the learning map.</p>

USE OF LINEAR ALGEBRA (IV)

Math 21b, Oliver Knill

	<p>MARKOV. Suppose we have three bags with 10 balls each. Every time we throw a dice and a 5 shows up, we move a ball from bag 1 to bag 2, if the dice shows 1 or 2, we move a ball from bag 2 to bag 3, if 3 or 4 turns up, we move a ball from bag 3 to bag 1 and a ball from bag 3 to bag 2. What distribution of balls will we see in average?</p>	<p>The problem defines a Markov chain described by a matrix $\begin{pmatrix} 5/6 & 1/6 & 0 \\ 0 & 2/3 & 1/3 \\ 1/6 & 1/6 & 2/3 \end{pmatrix}$. From this matrix, the equilibrium distribution can be read off as an eigenvector of a matrix.</p>
	<p>SPLINES In computer aided design (CAD) used for example to construct cars, one wants to interpolate points with smooth curves. One example: assume you want to find a curve connecting two points P and Q and the direction is given at each point. Find a cubic function $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ which interpolates.</p>	<p>If we write down the conditions, we will have to solve a system of 4 equations for four unknowns. Graphic artists (i.e. at the company "Pixar") need to have linear algebra skills also at many other topics in computer graphics.</p>
	<p>SYMBOLIC DYNAMICS Assume that a system can be in three different states a, b, c and that transitions $a \mapsto b, b \mapsto a, b \mapsto c, c \mapsto c, c \mapsto a$ are allowed. A possible evolution of the system is then $a, b, a, b, a, c, c, c, a, b, c, a, \dots$ One calls this a description of the system with symbolic dynamics. This language is used in information theory or in dynamical systems theory.</p>	<p>The dynamics of the system is coded with a symbolic dynamical system. The transition matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Information theoretical quantities like the "entropy" can be read off from this matrix.</p>
	<p>INVERSE PROBLEMS The reconstruction of a density function from projections along lines reduces to the solution of the Radon transform. Studied first in 1917, it is today a basic tool in applications like medical diagnosis, tokamak monitoring, in plasma physics or for astrophysical applications. The reconstruction is also called <i>tomography</i>. Mathematical tools developed for the solution of this problem lead to the construction of sophisticated scanners. It is important that the inversion $h = R(f) \mapsto f$ is fast, accurate, robust and requires as few data as possible.</p>	<p>Toy problem: We have 4 containers with density a, b, c, d arranged in a square. We are able to measure the light absorption by sending light through it. Like this, we get $o = a + b, p = c + d, q = a + c$ and $r = b + d$. The problem is to recover a, b, c, d. The system of equations is equivalent to $Ax = b$, with $x = (a, b, c, d)$ and $b = (o, p, q, r)$ and $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.</p>