

REVIEW I

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DEFINITIONS.

- Linear space** $\vec{0} \in X, \vec{x}, \vec{y} \in X, \lambda \in \mathbf{R} \Rightarrow \vec{x} + \vec{y} \in X, \lambda \vec{x} \in X$.
- Matrix** A is a $m \times n$ matrix, it has n columns and m rows, maps \mathbf{R}^n to \mathbf{R}^m .
- Square matrix** $n \times n$ matrix, maps \mathbf{R}^n to \mathbf{R}^n .
- Vector** $n \times 1$ matrix = column vector, $1 \times n$ matrix = row vector.
- Linear transformation** $T: \mathbf{R}^n \rightarrow \mathbf{R}^m, \vec{x} \mapsto A\vec{x}, T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}), T(\lambda\vec{x}) = \lambda T(\vec{x})$.
- Column vector** of A are images of standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$.
- Linear system of equations** $A\vec{x} = \vec{b}, m$ equations, n unknowns.
- Consistent system** $A\vec{x} = \vec{b}$: for every \vec{b} there is at least 1 solution \vec{x} .
- Inconsistent system** $A\vec{x} = \vec{b}$: there is no solution.
- Vector form of linear equation** $x_1\vec{v}_1 + \dots + x_n\vec{v}_n = \vec{b}, \vec{v}_i$ columns of A .
- Matrix form of linear equation** $\vec{w}_i \cdot \vec{x} = b_i, \vec{w}_i$ rows of A .
- Augmented matrix** of $A\vec{x} = \vec{b}$ is the matrix $[A|\vec{b}]$ which has one column more as A .
- Coefficient matrix** of $A\vec{x} = \vec{b}$ is the matrix A .
- Partitioned matrix**, where each entry is a matrix.
- Matrix multiplication** $[AB]_{ij} = \sum_k A_{ik}B_{kj}$, dot product of i -th row of A with j -th column of B .
- Gauss-Jordan elimination** $A \rightarrow \text{rref}(A)$ in row reduced echelon form.
- Gauss-Jordan elimination steps**: swapping rows, scaling rows, adding rows to other rows.
- Row reduced echelon form**: every nonzero row has leading 1, columns with leading 1 are 0 away from leading 1, every row with leading 1 has every rows above with leading 1 to the left.
- Pivot column** column with leading 1 in $\text{rref}(A)$.
- Rank of matrix** A . Number of leading 1 in $\text{rref}(A)$. It is equal to $\dim(\text{im}(A))$.
- Nullity of matrix** A . Is defined as $\dim(\ker(A))$.
- Kernel of linear transformation** $\{\vec{x} \in \mathbf{R}^n, A\vec{x} = \vec{0}\}$.
- Image of linear transformation** $\{A\vec{x}, \vec{x} \in \mathbf{R}^n\}$.
- Inverse Linear transformation** satisfying $S(T(x)) = x = T(S(x))$. Corresponding matrix $B = A^{-1}$.
- Rotation in plane** $\vec{x} \mapsto A\vec{x}, A = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$, counterclockwise rotation by angle α .
- Dilation in plane** $\vec{x} \mapsto \lambda\vec{x}$, also called scaling.
- Rotation-Dilation** $\vec{x} \mapsto A\vec{x}, A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Scale by $\sqrt{a^2 + b^2}$, rotate by $\arctan(b/a)$.
- Horizontal and vertical shear** $\vec{x} \mapsto A\vec{x}, A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \vec{x} \mapsto A\vec{x}, A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$.
- Shear** T leaves \vec{v} invariant and $A\vec{x} - \vec{x}$ parallel to \vec{v} . Shear-Check: all $A\vec{x} - \vec{x}$ are parallel.
- Reflection at line** $\vec{x} \mapsto A\vec{x}, A = \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix}$.
- Projection onto line containing unit vector** $T(\vec{x}) = (\vec{x} \cdot \vec{v})\vec{v}$.
- $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ **span** X : Every $\vec{x} \in X$ can be written as $\vec{x} = \sum_i a_i \vec{v}_i$.
- $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ **linear independent** X : $\vec{0} = \sum_i a_i \vec{v}_i \Rightarrow a_1 = \dots = a_n = 0$.
- $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ **basis** in X : linear independent in X and span X .
- dimension of linear space** X number of elements in a basis of X .
- B **coordinates** $[\vec{v}]_B = S^{-1}\vec{v}$, where $S = [\vec{v}_1, \dots, \vec{v}_n]$ contains basis vectors \vec{v}_i as columns.
- B **matrix** of T The matrix is $B = S^{-1}AS$.

RESULTS.

- Linear transformations.** T is linear: $T(\vec{0}) = \vec{0}, T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y}), T(\lambda\vec{x}) = \lambda T(\vec{x})$
- Solution.** A linear system of equations has either exactly 1, no or infinitely many solutions.
- Dimension formula.** $\dim(\ker(A)) + \dim(\text{im}(A)) = n$, where A is $m \times n$ matrix.
- Behavior of kernel under elimination** kernel stays invariant under Gauss-Jordan elimination.
- Behavior of image under elimination** image in general changes during Gauss-Jordan elimination.
- Basis of image** of A pivot columns of A form a basis of the image of A .
- Basis of kernel** of A introduce free variables for each non-Pivot column of A .
- Inverse** of 2×2 matrix switch diagonal elements, change sign of wings and divide by determinant.
- Kernel of composition** kernel of A is contained in the kernel of BA .
- Image of composition** image of BA is contained in the image of B .
- Matrix algebra** $(AB)^{-1} = B^{-1}A^{-1}, A(B + C) = AB + AC$, etc. Note: $AB \neq BA$ in general.
- A invertible** $\Leftrightarrow \text{rref}(A) = I \Leftrightarrow$ columns form basis $\Leftrightarrow \text{rank}(A) = n, \Leftrightarrow \text{nullity}(A) = 0$.

PROBLEM You know $B = \text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find the nullity and rank of A . Can you find the kernel of A ? Can you find the image of A .

SOLUTION. We can find the kernel but not the image. The dimensions of the image and kernel are 2 (2 pivot, 2 nonpivot columns). The linear system to B is $x + 2y + 5z = 0, z + 3u = 0$. Solving gives $u = t, z = -3t, y = s, x = -2s - 5t$, so that a general kernel element is $t[-5, 0, -3, 1] + s[-2, 1, 0, 0]$.

PROBLEM Given the basis $\vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ in the plane. a) Find the coordinates of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in this basis. b) Given a vector with coordinates $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in that basis, find the coordinates in the standard basis.

SOLUTION. $S = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$ has the inverse $S^{-1} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$. a) The vector has new coordinates $S^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -12 \\ 7 \end{bmatrix}$, b) The vector has the standard coordinates $S \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$.

PROBLEM Let A be a shear along a line L . Find $\ker(A - I_2), \text{im}(A - I_2)$ and $(A - I_2)^2$.

SOLUTION. Shears have the property that $(A - I)x = Ax - x$ is parallel to L , therefore $\text{im}(A - I)$ is L . The kernel of $A - I$ consists of vectors $Ax = x$. Because every $v \in L$ has this property, the kernel is L too. Both have dimension 1. We have $(A - I)^2 = 0$ because the image of $A - I$ is a subset of the kernel.

PROBLEM Let L be the line spanned by $\vec{v} = (1/2, 1/2, 0)$ and let T be the counterclockwise rotation about an angle $\pi/2$ around L (this means $\pi/2$ clockwise if you look from \vec{v} to the origin). Find the matrix A .

SOLUTION. Draw a good picture. \vec{e}_1 goes to $[1/2, 1/2, 1/\sqrt{2}]$, \vec{e}_2 goes to $[1/2, 1/2, -1/\sqrt{2}]$, \vec{e}_3 goes to $[-1/\sqrt{2}, 1/\sqrt{2}, 0]$. These are the columns of A , so that $A = \begin{bmatrix} 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$.

PROBLEM. Let A be a 3×3 matrix satisfying $A^2 = 0$. Show that the image of A is a subset of the kernel of A and determine all possible values for $\text{rank}(A)$. Give an example.

SOLUTION. If x is in the image then $x = Ay$ and $Ax = AAy = 0$ so that x is in the kernel. A is not invertible and can not have rank 3. It can be the 0 matrix with rank 0. It can have rank 1: take 0 in the first two columns and \vec{e}_1 in the last. It can not have rank 2 because the dimension of the kernel would be 1 and could not contain the kernel.

PROBLEM. Let $A = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$ Find B satisfying $B^2 = A$, determine $\text{rank}(B)$ and B^{17} .

SOLUTION. A is a rotation-dilation, a composition of a rotation by $\pi/2$ and dilation by 4. Take B as a rotation dilation with angle $\pi/4$ and dilation factor 2. The rank of B is 2 because if it were smaller, then also the rank of A were smaller. B^{17} is a rotation dilation with angle $17\pi/2 \sim \pi/2$ and dilation factor 2^{17} .

PROBLEM. If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in \mathbf{R}^n are linearly independent, are $\vec{w}_1 = \vec{v}_1, \vec{w}_2 = \vec{v}_1 + \vec{v}_2, \vec{w}_3 = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ also linearly independent?

SOLUTION. Yes: the linear map which maps the span of \vec{v}_i into the span of \vec{w}_i is invertible: it has the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

PROBLEM. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & \dots & 100 \\ 101 & 102 & 103 & \dots & 200 \\ \dots & \dots & \dots & \dots & \dots \\ 901 & 902 & 903 & \dots & 1000 \end{bmatrix}$.

SOLUTION. Deleting the first row from each other row makes the lower 99 rows all linearly dependent. The rank is 2.